

Unitarity in “quantization commutes with reduction”

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Abstract

Let M be a compact Kähler manifold equipped with a Hamiltonian action of a compact Lie group G . In this paper, we study the geometric quantization of the symplectic quotient $M//G$. Guillemin and Sternberg [Invent. Math. **67** (1982), 515–538] have shown, under suitable regularity assumptions, that there is a natural invertible map between the quantum Hilbert space over $M//G$ and the G -invariant subspace of the quantum Hilbert space over M .

Reproducing other recent results in the literature, we prove that in general the natural map of Guillemin and Sternberg is not unitary, *even to leading order in Planck’s constant*. We then modify the quantization procedure by the “metaplectic correction” and show that in this setting there is still a natural invertible map between the Hilbert space over $M//G$ and the G -invariant subspace of the Hilbert space over M . We then prove that this modified Guillemin–Sternberg map is asymptotically unitary to leading order in Planck’s constant. The analysis also shows a good asymptotic relationship between Toeplitz operators on M and on $M//G$.

Keywords: geometric quantization, symplectic reduction, semiclassical limit, Toeplitz operators

1 Introduction

Let M be an integral compact Kähler manifold with symplectic form ω . Following the program of geometric quantization, suppose we are given a Hermitian holomorphic line bundle ℓ with connection over M , chosen in such a way that the curvature of ℓ is equal to $-i\omega$. We then consider $\ell^{\otimes k}$, the k th tensor power of ℓ , where in this setting we interpret k as the reciprocal of Planck’s constant \hbar . The Hilbert space of quantum states associated to M is then (for a fixed value of $\hbar = 1/k$) the space of holomorphic sections of $\ell^{\otimes k}$.

Now suppose that we are given a Hamiltonian action of a connected compact Lie group G on M . Then we can construct the symplectic quotient $M//G$, which is another compact Kähler manifold (under suitable regularity assumptions on the action of G). The line bundle ℓ naturally descends to a bundle $\hat{\ell}$ over $M//G$, and the quantum Hilbert space associated to $M//G$ is then the space of holomorphic sections of $\hat{\ell}^{\otimes k}$. This space of sections is what we obtain by performing reduction by G *before* quantization. Alternatively, we may *first* perform quantization and *then* perform reduction at the quantum level, which amounts to restricting to the space of G -invariant sections of $\ell^{\otimes k}$.

A classic result of Guillemin and Sternberg is that there is a natural invertible linear map between the “first reduce and then quantize” space (the space of all holomorphic sections of the bundle $\hat{\ell}^{\otimes k}$ over $M//G$) and the “first quantize and then reduce” space (the space of G -invariant holomorphic sections of the bundle $\ell^{\otimes k}$ over M). This result is sometimes described as saying that quantization commutes with reduction. However, from the point of view of quantum mechanics, it is not just the vector space structure of the quantum Hilbert space that is important, but also the inner product.

It is natural, then, to investigate the extent to which the Guillemin–Sternberg map is unitary. Some work in this direction has already been done, and in general, it has been found that the

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obstruction to unitarity is a certain function on $M//G$ (sometimes referred to as the effective potential). This obstruction, in some form, was identified independently in the works of [Flu98], [Char06b], and [Pao05], and [MZ06]. See also [Got86], [Hal02], and [Hue06]. We discuss the relation of our work to these in Section 1.2.

Our first main result is a new proof (similar to that of Charles in [Char06b] for the torus case) that the Guillemin–Sternberg map is not unitary in general, and indeed that the map does not become asymptotically unitary as $k \rightarrow \infty$ (i.e., as $\hbar \rightarrow 0$). We show that the obstruction to asymptotic unitarity of the map is the volume of the G -orbits inside the zero-set of the moment map in M . If these G -orbits do not all have the same volume, then the Guillemin–Sternberg map will not be asymptotically unitary. This failure of asymptotic unitarity is troubling from a physical point of view. After all, although one expects different quantization procedures (e.g., performing quantization and reduction in different orders) to give different results, one generally regards these differences as “quantum corrections” that should disappear as \hbar tends to zero.

The main contribution of this paper is a solution to this unsatisfactory situation: we include the so-called metaplectic correction, which involves tensoring the original line bundle with the square root of the canonical bundle (assuming that such a square root exists). We first show that one can define a natural map of Guillemin–Sternberg type in the presence of the metaplectic correction, and this natural map is invertible for all sufficiently large values of the tensor power k . We then show that this modified Guillemin–Sternberg map, unlike the original one, is asymptotically unitary in the limit as k tends to infinity.

In the rest of this introduction, we describe our results in greater detail and compare them to previous results of other authors.

1.1 Main results

Let M^{2n} be a compact Kähler manifold with symplectic form ω , complex structure J , and Riemannian metric $B = \omega(\cdot, J\cdot)$. Assume that M is quantizable, that is, that the class $[\omega/2\pi]$ is integral. Choose a Hermitian line bundle ℓ over M with compatible connection ∇ in such a way that the curvature of ∇ is $-i\omega$. (Such a bundle exists because M is quantizable.) We call ℓ a prequantum bundle for M . We denote the Hermitian form on ℓ , which we take to be linear in the second factor, by (\cdot, \cdot) and we denote the pointwise magnitude of a section s of ℓ by $|s|^2(x) = (s, s)(x)$. The connection and Hermitian form on ℓ induce a connection and Hermitian form on $\ell^{\otimes k}$ which we denote by the same symbols.

The complex structure on M gives $\ell^{\otimes k}$ the structure of a holomorphic line bundle in which the holomorphic sections are those that are covariantly constant in the $(0, 1)$ -directions. We let $\mathcal{H}(M; \ell^{\otimes k})$ denote the (finite-dimensional) space of holomorphic sections of $\ell^{\otimes k}$. The symplectic form on M induces a volume form

$$\varepsilon_\omega := \frac{\omega^{\wedge n}}{n!}, \quad (1.1)$$

which is known as the Liouville volume form. We make $\mathcal{H}(M; \ell^{\otimes k})$ into a Hilbert space by considering the inner product given by

$$\langle s_1, s_2 \rangle := (k/2\pi)^{n/2} \int_M (s_1, s_2) \varepsilon_\omega. \quad (1.2)$$

In geometric quantization, we interpret the tensor power k as the reciprocal of Planck’s constant \hbar . Thus the study of holomorphic sections of $\ell^{\otimes k}$ in the limit $k \rightarrow \infty$, familiar from algebraic geometry, is in this setting interpreted as the “semiclassical limit” (i.e., the limit $\hbar \rightarrow 0$). The semiclassical limit in geometric quantization has garnered much recent interest. See for example the work of Borthwick and Uribe [BU96] and [BU00] on a symplectic version of the Kodaira embedding theorem and the work of Borthwick, Paul, and Uribe [BPU95] on applications to relative Poincaré series.

Suppose now that we are given a Hamiltonian action of a connected compact Lie group G of dimension d , together with an equivariant moment map Φ for this action. We assume (as in [GS82]) that 0 is in the image of Φ , that 0 is a regular value of Φ , and that G acts freely on $\Phi^{-1}(0)$. The symplectic quotient $M//G$ is then defined to be

$$M//G = \Phi^{-1}(0)/G.$$

The quotient $M//G$ acquires from M the structure of a quantizable Kähler manifold of dimension $2(n-d)$; in particular the symplectic form ω on M descends to a symplectic form $\widehat{\omega} \in \Omega^2(M//G)$. We assume that the action of G lifts to ℓ . Then ℓ descends to a holomorphic Hermitian line bundle $\widehat{\ell}$ with connection over $M//G$, with curvature $-i\widehat{\omega}$. Let $\mathcal{H}(M//G; \widehat{\ell}^{\otimes k})$ denote the space of holomorphic sections of $\widehat{\ell}^{\otimes k}$ and use on this space the inner product given by

$$\langle s_1, s_2 \rangle := (k/2\pi)^{(n-d)/2} \int_{M//G} (s_1, s_2) \varepsilon_{\widehat{\omega}}. \quad (1.3)$$

We are interested in the relationship between two different Hilbert spaces: first, the space of G -invariant holomorphic sections of $\ell^{\otimes k}$ over M , with the inner product (1.2); and second, the space of all holomorphic sections of $\widehat{\ell}^{\otimes k}$ over $M//G$, with the inner product (1.3). The first Hilbert space, denoted $\mathcal{H}(M; \ell^{\otimes k})^G$, is the one obtained by first quantizing and then reducing by G ; the second one is obtained by first reducing by G and then quantizing. According to Guillemin and Sternberg [GS82], there is (for each k) a natural one-to-one and onto linear map A_k from $\mathcal{H}(M; \ell^{\otimes k})^G$ onto $\mathcal{H}(M//G; \widehat{\ell}^{\otimes k})$. In particular, these two spaces have the same dimension.

The map A_k is defined in the only reasonable way: one takes a G -invariant holomorphic section s of $\ell^{\otimes k}$, restricts it to $\Phi^{-1}(0)$, and then lets s descend from $\Phi^{-1}(0)$ to $\Phi^{-1}(0)/G$. From the way that the complex structure on $M//G$ is defined, it is easy to see that A_k maps holomorphic sections of $\ell^{\otimes k}$ to holomorphic sections of $\widehat{\ell}^{\otimes k}$. The hard part is to show that this map is invertible, i.e., that a section of $\widehat{\ell}^{\otimes k}$, after being lifted to $\Phi^{-1}(0)$, can be extended holomorphically to a (G -invariant) section over all of M . In proving this, Guillemin and Sternberg make use of a holomorphic action of the “complexified” group $G_{\mathbb{C}}$ on M .

We wish to determine the extent to which the map A_k is unitary. We show that (for each k) there exists a function I_k on $M//G$ with the property that for each G -invariant holomorphic section s of $\ell^{\otimes k}$ we have

$$\|s\|^2 = (k/2\pi)^{n/2} \int_M |s|^2 \varepsilon_{\omega} = (k/2\pi)^{(n-d)/2} \int_{M//G} |A_k s|^2 I_k \varepsilon_{\widehat{\omega}}.$$

Our first main result is the following.

Theorem 1.1. *The functions I_k satisfy*

$$\lim_{k \rightarrow \infty} I_k([x_0]) = 2^{-d/2} \text{vol}(G \cdot x_0)$$

for all $x_0 \in \Phi^{-1}(0)$, and the limit is uniform. Here $\text{vol}(G \cdot x_0)$ denotes the volume of the G -orbit of x_0 with respect to the Riemannian structure inherited from M .

If all of the G -orbits in the zero-set of the moment map have the same volume, then Theorem 5.1 implies that the natural map A_k is asymptotically a constant multiple of a unitary map in the limit as \hbar tends to zero. If, as is usually the case, the G -orbits in the zero-set do not all have the same volume, then using peaked sections, we can show that A_k is not asymptotically a constant multiple of a unitary map, in a sense described in Theorem 5.3.

Now, it is probably unreasonable to expect the natural map between the “first reduce and then quantize” space and the “first quantize and then reduce” space to be unitary. Different ways of performing quantization (e.g., before reduction and after reduction) in general give inequivalent

results, especially if equivalence is measured by the existence of a geometrically natural *unitary* (as opposed to merely invertible) map. On the other hand, one expects differences between the procedures to be “quantum corrections” that vanish for small \hbar . The situation reflected in Theorem 1.1, then, is unsatisfactory; the geometrically natural map between $\mathcal{H}(M; \ell^{\otimes k})^G$ and $\mathcal{H}(M//G; \hat{\ell}^{\otimes k})$ is not unitary *even to leading order in \hbar* .

To remedy this situation, we introduce the “metaplectic” or “half-form” correction. There are various results which seem to indicate the geometric quantization works better with half-forms (indeed, our main results can be seen as further justification of this claim). For example, it is only with half-forms that the quantization of the simple harmonic oscillator has the correct zero-point energy (see [Woo91, Chap. 10] for details and further examples). For this reason, we might expect (and it is indeed the case) that the inclusion of half-forms remedies the unsatisfactory situation described by Theorem 1.1.

The metaplectic correction consists of tensoring $\ell^{\otimes k}$ with a square root of the canonical bundle K of M , assuming such a square root exists (equivalently, the vanishing of the second Stiefel–Whitney class), and similarly tensoring $\hat{\ell}^{\otimes k}$ with a square root of the canonical bundle \hat{K} of $M//G$. Assuming that the G -action lifts to $\ell^{\otimes k} \otimes \sqrt{K}$, our second main result is an analog of [GS82, Thm 5.2] in the presence of half-forms.

Theorem 1.2. *For each k , there is a natural linear map B_k between the space of G -invariant holomorphic sections of $\ell^{\otimes k} \otimes \sqrt{K}$ over M and the space of all holomorphic sections of $\hat{\ell}^{\otimes k} \otimes \sqrt{\hat{K}}$ over $M//G$. Furthermore, this map is invertible for all sufficiently large k .*

We then show that there exists a function J_k on $M//G$ such that for each G -invariant holomorphic section r of $\ell^{\otimes k} \otimes \sqrt{K}$ we have

$$\|r\|^2 := (k/2\pi)^{n/2} \int_M |r|^2 \varepsilon_\omega = (k/2\pi)^{(n-d)/2} \int_{M//G} |B_k r|^2 J_k \varepsilon_{\hat{\omega}}.$$

Our last main result is the following, which implies that the maps B_k , unlike the maps A_k , are asymptotically unitary in the limit $k \rightarrow \infty$ (i.e., $\hbar \rightarrow 0$).

Theorem 1.3. *The functions J_k satisfy*

$$\lim_{k \rightarrow \infty} J_k([x_0]) = 1$$

for all $x_0 \in \Phi^{-1}(0)$, and the limit is uniform.

The origin of the unwanted volume factor in Theorem 1.1 is the relationship between the volume measure on $\Phi^{-1}(0)$ and the volume measure on $\Phi^{-1}(0)/G$ (Lemma 4.5). The metaplectic correction introduces a compensating volume factor in the pointwise magnitude of a section over $\Phi^{-1}(0)$ and the pointwise magnitude of the corresponding section over $\Phi^{-1}(0)/G$ (Theorem 3.3).

The proofs of our main results make use of a holomorphic action of the “complexified” group $G_{\mathbb{C}}$ on M , obtained in [GS82] by analytic continuation of the action of G . We let M_s denote the **stable set**, that is, the set of points in M that can be moved into the zero-set of the moment map by the action of $G_{\mathbb{C}}$. The stable set is an open set of full measure in M and the symplectic quotient $M//G = \Phi^{-1}(0)/G$ can be identified naturally with $M_s/G_{\mathbb{C}}$. We show how the magnitude of a G -invariant section (of $\ell^{\otimes k}$ or $\ell^{\otimes k} \otimes \sqrt{K}$) varies in a predictable way as one moves off of $\Phi^{-1}(0)$ along a $G_{\mathbb{C}}$ -orbit; this fact, which is crucial to our analysis, has been noticed before by Guillemin and Sternberg [GS82] and by Donaldson [Don04]. The densities I_k and J_k are obtained by integrating certain quantities related to the moment map over each $G_{\mathbb{C}}$ -orbit. The asymptotic behavior of these densities is then determined by Laplace’s method (sometimes referred to as the stationary phase approximation or the method of steepest descent).

In proving the invertibility of our Guillemin–Sternberg-type map B_k , we first show that a G -invariant section of $\ell^{\otimes k} \otimes \sqrt{K}$ defined over $\Phi^{-1}(0)$ can be extended to a holomorphic, $G_{\mathbb{C}}$ -invariant

section over the stable set M_s . We then show that for *all sufficiently large* k , these sections have a removable singularity over the complement of the stable set and thus extend to holomorphic sections over all of M . Here, the presence of half-forms makes the situation slightly more complicated than in [GS82], where the natural map A_k is shown to be invertible for all k .

A trivial modification of the arguments described above yields a formula that relates a Toeplitz operator with a G -invariant symbol upstairs, restricted to the space of G -invariant sections, to a certain Toeplitz operator downstairs. Suppose f is a smooth, G -invariant function on M . Then in the case with half-forms, the result is that the Toeplitz operator with symbol f upstairs, restricted to the G -invariant subspace, is asymptotically equivalent to the Toeplitz operator with symbol \hat{f} downstairs, where \hat{f} is obtained by restricting f to $\Phi^{-1}(0)$ and letting it descend to $\Phi^{-1}(0)/G$.

For the reader's convenience, we collect here the various assumptions made in this paper. We begin with a compact Kähler manifold M of real dimension $2n$, with symplectic form ω . We assume that $[\omega/2\pi]$ is an integral cohomology class. We assume that M is equipped with a holomorphic and Hamiltonian action of a compact d -dimensional Lie group G with equivariant moment map $\Phi : M \rightarrow \mathfrak{g}^*$. We assume that 0 is in the image of the moment map, that 0 is a regular value of the moment map, and that G acts freely on the zero-set $\Phi^{-1}(0)$. Since the symplectic form is integral, there exists over M at least one complex Hermitian line bundle with compatible connection with curvature $-i\omega$; we fix some choice and denote it by ℓ . In a canonical way, the moment map defines an infinitesimal holomorphic action of G on ℓ . We assume that this infinitesimal action can be exponentiated. These assumptions are the same as in [GS82] and are all the assumptions relevant to Theorem 1.1.

We need additional assumptions for Theorems 1.2 and 1.3. Let $K = \bigwedge^n (T^*M)^{1,0}$ denote the canonical bundle of M . We assume that K admits a square root $\sqrt{K} \rightarrow M$ (this is equivalent to the vanishing of the second Stiefel-Whitney class of M), and finally we assume that the G -action on K (which is induced from that on M) can be lifted to an action on \sqrt{K} .

1.2 Prior results

To the best of our knowledge, the first work that systematically address the question of unitarity in the context of quantization and reduction of Kähler manifolds is the Ph.D. thesis of Flude [Flu98]. Flude gives a formal computation of the leading-order asymptotics of the density I_k as k tends to infinity, without the metaplectic correction. Flude's main computation is essentially the same as ours (in the case without the metaplectic correction): an application of Laplace's method. However, Flude's result is not rigorous, because he considers the magnitude of an invariant section only in a small neighborhood of the zero-set. This should be compared to our Theorem 4.1, which gives an expression for the magnitude of an invariant section that is valid everywhere. (Even after Theorem 4.1 is obtained, some control is required over the blow-up of certain Jacobians near the unstable set.) Nevertheless, Flude does identify the volume of the G -orbits in $\Phi^{-1}(0)$ as the obstruction to the asymptotic unitarity of the Guillemin–Sternberg map.

Next, there is the work of L. Charles [Char06b], who considers the case in which the group G is a torus and without half-forms. (Charles does consider half-forms in a related context in [Char06a].) In the torus case, Proposition 4.17 of [Char06b], with f identically equal to 1 in a neighborhood of the zero-set, is essentially the same as our Theorem 4.2. Proposition 4.18 of [Char06b] is then essentially the same as the first part of our Theorem 5.1. We do, however, go slightly beyond the approach of Charles, in that we give an *exact* (not just asymptotic to all orders) expression for the norm of an invariant section upstairs as an integral over the downstairs manifold. This requires some control over the blow-up of certain Jacobians as one approaches the unstable set.

Then there is the work of Paoletti [Pao05]. He takes a different approach to measuring unitarity, looking at the behavior of orthonormal bases. Nevertheless, his result agrees with ours (in the case without half-forms) in that he identifies the volume of the G -orbits in the zero-set of the moment map as an obstruction to unitarity of the Guillemin–Sternberg map.

Finally, there is the work of Ma and Zhang in [MZ06]. In this paper, the authors compute an asymptotic expansion of the equivariant Bergman kernel, and our Theorem 1.1 is a special case of their Theorem 0.1. In contrast to our work and the works of Flude and Charles (all of which are based on Laplace’s method), those of Paoletti and Ma–Zhang are based on the microlocal analysis developed by L. Boutet de Monvel and Guillemin in [BdMG81].

To the best of our knowledge, ours is the first paper that constructs a Guillemin–Sternberg-type map in the presence of the metaplectic correction and therefore the first to show in a general setting that the metaplectic correction improves the situation in regard to unitarity. (Such an improvement had already been suggested by the special examples considered in [DH99] and [Hal02]. See the last section for a discussion of these examples.)

We close this section by mentioning the work of Gotay [Got86], who considers the relationship between quantization and reduction in the context of a cotangent bundle equipped with the vertical polarization (a *real* polarization). Gotay considers the cotangent bundle of an n -manifold Q and assumes that a compact group G of dimension d acts freely on Q . The induced action of G on T^*Q is then free and Hamiltonian. Gotay includes half-forms in the quantization (as one must in the case of the vertical polarization) and obtains a *unitary* map between the quantize-then-reduce space and the reduce-then-quantize space.

Gotay’s work does not overlap with ours, because we consider only complex polarizations. Nevertheless, it is worth noting that Gotay obtains *exact* unitarity, whereas we obtain unitarity only asymptotically as Planck’s constant tends to zero, even with half-forms. The reason that the cotangent bundle case is nicer than the Kähler case seems to be that there is less differential geometry involved. There is no additional structure on the manifold Q (metric or measure or complex structure) that enters. Gotay’s unitarity result comes down to the relationship between the integral over Q of a G -invariant n -form α and the integral over Q/G of an $(n - d)$ -form $\hat{\alpha}$ obtained by contracting α with the generators of the G -action and then letting the result descend to Q/G . (Here α is the square of a half-form.) In the Kähler case, we have a mapping between $(n, 0)$ -forms on M and $(n - d, 0)$ -forms on $M//G$ defined in a manner very similar to that in [Got86]. However, it does not make sense to integrate an $(n, 0)$ -form over M , because M has real dimension $2n$. So the norm of a half-form cannot be computed by simply squaring and integrating. Rather, one uses a more complicated procedure involving both the complex structure on M and the Liouville volume measure. The relationship between the norm upstairs and the norm downstairs is correspondingly more complicated and involves the geometric structures that we have on M but not on Q .

Another way of thinking about the Kähler case is to observe that the way one lifts a section downstairs to a section upstairs is by lifting to a $G_{\mathbb{C}}$ -invariant (not just G -invariant) section upstairs. However, the action of $G_{\mathbb{C}}$ preserves only the complex structure on M and not the symplectic structure on M or the Hermitian structure on the relevant line bundles. Thus the relationship between the upstairs norm and the downstairs norm involves the way the volume measure and the magnitude of an invariant section vary over each $G_{\mathbb{C}}$ -orbit. In the cotangent bundle case, by contrast, only a G -action is involved and this preserves all the relevant structure.

2 Preliminaries

We begin this section by recalling the method of geometric quantization (at the moment, without half-forms) as it applies to Kähler manifolds. We then recall the notion of the Marsden–Weinstein or symplectic quotient and explain the special form this construction takes in the setting of Kähler manifolds. Finally, we describe the natural invertible map, due to Guillemin and Sternberg, between the “quantize then reduce” space and the “reduce then quantize” space.

2.1 Kähler quantization

Let $(M, \omega, J, B = \omega(\cdot, J\cdot))$ be a Kähler manifold with symplectic form ω , complex structure J , and Riemannian metric $B = \omega(\cdot, J\cdot)$. We assume that M is connected, compact, and integral (i.e., that $[\omega/2\pi]$ is an integral cohomology class). We fix once and for all a Hermitian line bundle ℓ with compatible connection ∇ , chosen in such a way that the curvature of ∇ is equal to $-i\omega$. The connection on ℓ induces a connection on the k th tensor power $\ell^{\otimes k}$, also denoted ∇ . For any k , $\ell^{\otimes k}$ may be given the structure of a holomorphic line bundle in such a way that the holomorphic sections are precisely those that are covariantly constant in the $(0, 1)$ (or \bar{z}) directions.

We interpret the tensor power k as the reciprocal of Planck's constant \hbar . For each fixed value of k (or \hbar), the quantum Hilbert space is the space of holomorphic sections of $\ell^{\otimes k}$, denoted $\mathcal{H}(M; \ell^{\otimes k})$. If ε_ω denotes the Liouville volume form (1.1), then we use the following natural inner product on $\mathcal{H}(M; \ell^{\otimes k})$,

$$\langle s_1, s_2 \rangle := (k/2\pi)^{n/2} \int_M (s_1, s_2) \varepsilon_\omega,$$

as in (1.2), where (s_1, s_2) is the (pointwise) Hermitian structure on $\ell^{\otimes k}$.

2.2 Kähler reduction

We now assume that we are given a smooth action of a connected compact Lie group G of dimension d on M . We assume that the action preserves all of the structure (symplectic, complex, Riemannian) of M . Let \mathfrak{g} denote the Lie algebra of G and for each $\xi \in \mathfrak{g}$, let X^ξ denote the vector field describing the infinitesimal action of ξ on M . That is, $X^\xi(x) = \frac{d}{dt} e^{t\xi} \cdot x|_{t=0}$.

We assume that for each $\xi \in \mathfrak{g}$ there exists a smooth function ϕ_ξ on M such that X^ξ is the Hamiltonian vector field associated to ϕ_ξ ; i.e., such that

$$i(X^\xi)\omega = d\phi_\xi.$$

(This is automatically the case if G is semisimple or if M is simply connected.) For each ξ , the function ϕ_ξ is unique up to a constant. Since M is compact, it is always possible to choose the constants in such a way that the map $\xi \rightarrow \phi_\xi$ is linear and that $\{\phi_\xi, \phi_\eta\} = -\phi_{[\xi, \eta]}$, and we fix one particular choice of constants with these two properties. (For any G , one way to choose the constants with these two properties is to require each ϕ_ξ to have integral zero over M . If G is semisimple, then these two properties uniquely determine the choice of the constants.)

We may put together the functions ϕ_ξ into a “moment map” $\Phi : M \rightarrow \mathfrak{g}^*$ given by

$$\Phi(x)(\xi) = \phi_\xi(x)$$

for each $\xi \in \mathfrak{g}$ and $x \in M$. The condition $\{\phi_\xi, \phi_\eta\} = -\phi_{[\xi, \eta]}$ ensures that the moment map is equivariant with respect to the action of G on M and the coadjoint action of G on \mathfrak{g}^* . We assume that 0 is in the image of the moment map and that it is a regular value, so that the zero-set $\Phi^{-1}(0)$ is a submanifold of M . We assume, moreover, that G acts freely on $\Phi^{-1}(0)$ so that the quotient is a manifold. The quotient

$$M//G := \Phi^{-1}(0)/G$$

is then called the symplectic or Marsden–Weinstein [MW74] quotient of M by G .

The quotient $M//G$ inherits a symplectic structure from M : there is a unique symplectic form $\widehat{\omega} \in \Omega^2(M//G)$ such that $i^*\omega = \pi^*\widehat{\omega}$, where $i : \Phi^{-1}(0) \rightarrow M$ denotes the inclusion map and $\pi : \Phi^{-1}(0) \rightarrow \Phi^{-1}(0)/G$ is the quotient map. Furthermore, $[\widehat{\omega}/2\pi]$ is an integral cohomology class on $M//G$.

So far, we have described how the symplectic structure on M induces a symplectic structure on $M//G$. We now show how the complex structure on M induces a complex structure on $M//G$. The descent of the complex structure can be understood either “infinitesimally” (by describing

how the distribution of $(1, 0)$ -vectors descends) or “globally” (by realizing $M//G$ as the quotient of the “stable set” in M by the complexification of G).

We begin with the infinitesimal approach. Let $T^{1,0}M$ be the Kähler polarization on M ; i.e., $T^{1,0}M$ is the n -dimensional (complex) distribution consisting of type- $(1, 0)$ vector fields on M . For future use, we note that the projection $\pi_+ : T^{\mathbb{C}}M \rightarrow T^{1,0}M$ is given by

$$\pi_+ X = \frac{1}{2}(1 - iJ)X.$$

In [GS82], Guillemin and Sternberg show that the G -orbits in the zero-set are totally real submanifolds; i.e., for all $x_0 \in \Phi^{-1}(0)$,

$$T_{x_0}^{1,0}M \cap \{X_{x_0}^\xi : \xi \in \mathfrak{g}\} = \{0\}.$$

Moreover, they show that $T^{\mathbb{C}}(\Phi^{-1}(0)) \cap T^{1,0}M$ is a G -invariant complex distribution of complex rank $n - d$, and that $\pi_*(T^{\mathbb{C}}(\Phi^{-1}(0)) \cap T^{1,0}M)$ is a well-defined integrable complex distribution of rank $n - d$ on $M//G$. In fact, the complex structure induced by this distribution defines a Kähler structure on $M//G$, and we henceforth make the identification

$$T^{1,0}(M//G) = \pi_*(T^{\mathbb{C}}(\Phi^{-1}(0)) \cap T^{1,0}M). \quad (2.1)$$

We now turn to the global approach, which is more intuitive and much more useful for the computations we will carry out in this paper. We have assumed that the action of G is holomorphic. Using this, it can be shown [GS82] that the action of G can be analytically continued to a holomorphic action of the “complexified” group $G_{\mathbb{C}}$ on M ; the infinitesimal action of this continuation is defined by

$$X^{i\xi} := JX^\xi, \quad \xi \in \mathfrak{g}. \quad (2.2)$$

Here $G_{\mathbb{C}}$ is a connected complex Lie group containing G as a maximal compact subgroup, and the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ is the complexification of \mathfrak{g} . The Cartan decomposition is a diffeomorphism $G_{\mathbb{C}} \simeq \exp(i\mathfrak{g})G$. Moreover, the set $\exp(i\mathfrak{g})$ is diffeomorphic to the vector space \mathfrak{g} , the diffeomorphism being the exponential map, and so as smooth manifolds we have $G_{\mathbb{C}} = \mathfrak{g} \times G$. See [Kna02, Sec. 6.3] for details.

We let the **stable set** M_s denote the saturation of the zero-set $\Phi^{-1}(0)$ by the action of $G_{\mathbb{C}}$:

$$M_s := G_{\mathbb{C}} \cdot \Phi^{-1}(0). \quad (2.3)$$

That is, M_s is the set of points in M that can be moved into $\Phi^{-1}(0)$ by the action of $G_{\mathbb{C}}$. (See (2.5) below for another characterization of the stable set.) The following properties of M_s follow from results in [GS82]: (1) M_s is an open set of full measure in M ; (2) $G_{\mathbb{C}}$ acts freely on M_s ; and (3) each $G_{\mathbb{C}}$ -orbit in M_s intersects $\Phi^{-1}(0)$ in precisely one G -orbit.

We will show shortly (Theorem 2.1) that M_s is in fact a principal $G_{\mathbb{C}}$ -bundle over $\Phi^{-1}(0)/G$. This implies (Corollary 2.2) that the action of $G_{\mathbb{C}}$ on M_s is proper. (Keep in mind that we are assuming that 0 is a regular value of the moment map and that G acts freely on $\Phi^{-1}(0)$.) Because the action of $G_{\mathbb{C}}$ is free, proper and holomorphic, the quotient $M_s/G_{\mathbb{C}}$ has the structure of a complex manifold. On the other hand, since each $G_{\mathbb{C}}$ -orbit in M_s intersects $\Phi^{-1}(0)$ in precisely one G -orbit, we have a natural bijective identification

$$\Phi^{-1}(0)/G = M_s/G_{\mathbb{C}}. \quad (2.4)$$

Once we know that $M_s/G_{\mathbb{C}}$ has the structure of a complex manifold, it is not hard to see that this complex structure agrees (under the above identification) with the one obtained infinitesimally in (2.1). (If $\pi_{\mathbb{C}} : M_s \rightarrow M_s/G_{\mathbb{C}}$ is the quotient map and we identify $M_s/G_{\mathbb{C}}$ with $\Phi^{-1}(0)/G$, then $\pi_{\mathbb{C}}$ agrees with π on $\Phi^{-1}(0)$. Thus $(\pi_{\mathbb{C}})_*$ agrees with π_* on vectors tangent to $\Phi^{-1}(0)$. But it is not

hard to show that every $(1, 0)$ -vector at a point in $\Phi^{-1}(0)$ is the sum of a $(1, 0)$ -vector tangent to $\Phi^{-1}(0)$ and a $(1, 0)$ -vector tangent to the $G_{\mathbb{C}}$ orbit.)

Although we do not require this result, we explain briefly how the quotient $M_s/G_{\mathbb{C}}$ can be identified with the quotient in geometric invariant theory (GIT). In GIT, there are several (in general, inequivalent) notions of stable points for the action of a complex reductive group (such as $G_{\mathbb{C}}$) on a compact Kähler manifold. (In [MFK94], these are called semistable, stable, and properly stable points.) Under our assumptions—that 0 is a regular value of the moment map and that G acts freely on $\Phi^{-1}(0)$ —these different notions of stability in GIT turn out to be equivalent to one another and to the notion of stability in (2.3). (As a consequence of a result of Kempf and Ness [KN79], the different notions of stability of $y \in M$ in GIT are equivalent to: (1) the closure of $G_{\mathbb{C}} \cdot y$ intersects $\Phi^{-1}(0)$, (2) $G_{\mathbb{C}} \cdot y$ itself intersects $\Phi^{-1}(0)$, and (3) $G_{\mathbb{C}} \cdot y$ intersects $\Phi^{-1}(0)$ and the stabilizer of y is finite. Condition 2 is our definition of the stable set. Since in our case M_s is open and acted on freely by $G_{\mathbb{C}}$, the other two conditions are equivalent to Condition 2.) The GIT quotient coincides, as a set and topologically, with $M_s/G_{\mathbb{C}}$. See for example Theorem 8.3 of [MFK94] or Section 3 of [HHL94].

The stable set can also be characterized in terms of the G -invariant holomorphic sections of $\ell^{\otimes k}$ (see for example Donaldson's work in [Don04], or [GS82, Theorem 5.5]):

$$M_s = \{x \in M : s(x) \neq 0 \text{ for some } s \in \mathcal{H}(M; \ell^{\otimes k})^G \text{ for some } k\}. \quad (2.5)$$

Suppose, then, that s is an element of $s \in \mathcal{H}(M; \ell^{\otimes k})^G$ that is not identically zero. (That such a section exists for some k follows from (2.5) and is established in [GS82, Appendix].) Then the unstable set $M_s \setminus M$ is contained in the zero-set of s and is therefore a set of codimension at least one. This implies, for example, that the unstable set has measure zero.

We now establish a structure result for M_s that implies, among other things, that the action of $G_{\mathbb{C}}$ on M_s is proper. Let $\pi_{\mathbb{C}} : M_s \rightarrow M//G$ be the complex quotient map under the identification of $M_s/G_{\mathbb{C}}$ with $M//G$. That is to say, $\pi_{\mathbb{C}}(z) = \pi(x)$ if x belongs to the unique G -orbit in $\Phi^{-1}(0)$ contained in the $G_{\mathbb{C}}$ -orbit of z .

Theorem 2.1. *The map $\pi_{\mathbb{C}} : M_s \rightarrow M//G$ is smooth (in particular, continuous), and M_s has the structure of a smooth principal $G_{\mathbb{C}}$ -bundle with respect to this map.*

Corollary 2.2. *The action of $G_{\mathbb{C}}$ on M_s is proper.*

Corollary 2.3. *The map $\Lambda : \mathfrak{g} \times \Phi^{-1}(0) \rightarrow M_s$ given by*

$$\Lambda(\xi, x) = e^{i\xi} \cdot x$$

is bijective and a diffeomorphism.

Proof of Corollary 2.2. Suppose we have $y_k \in M_s$ and $g_k \in G_{\mathbb{C}}$ such that y_k converges to $y \in M_s$ and such that $g_k \cdot y_k$ converges to $z \in M_s$. We will show that g_k must then be convergent, which implies that the action of $G_{\mathbb{C}}$ on M_s is proper [DK00, pp. 53 ff. and Chap. 2]. Note that $\pi_{\mathbb{C}}(g_k y_k) = \pi_{\mathbb{C}}(y_k)$. Taking limits and using the continuity of $\pi_{\mathbb{C}}$ gives that $\pi_{\mathbb{C}}(z) = \pi_{\mathbb{C}}(y)$. Choose a neighborhood U of $\pi_{\mathbb{C}}(z)$ such that $\pi_{\mathbb{C}}^{-1}(U)$ is homeomorphic to $G_{\mathbb{C}} \times U$ in such a way that the $G_{\mathbb{C}}$ action corresponds to left multiplication in $G_{\mathbb{C}}$. Under this homeomorphism, y_k corresponds to (h_k, u_k) and y corresponds to (h, u) , with $h_k \rightarrow h$ and $u_k \rightarrow u$. Then $g_k \cdot y_k$ corresponds to $(g_k h_k, u_k)$ and z corresponds to (f, u) , with $g_k h_k \rightarrow f$. Thus, g_k converges to $h^{-1}f$. \square

Proof of Corollary 2.3. The bijectivity of Λ follows from: (1) the freeness of the action of $G_{\mathbb{C}}$ on M_s , (2) the fact that each $G_{\mathbb{C}}$ -orbit intersects $\Phi^{-1}(0)$ in a single G -orbit, and (3) the bijectivity of the Cartan decomposition of $G_{\mathbb{C}}$ [Kna02, Thm 6.31, pp. 362]. To see that Λ is a diffeomorphism, decompose $\pi_{\mathbb{C}}^{-1}(U)$ as $G_{\mathbb{C}} \times U$. Since the Cartan decomposition of $G_{\mathbb{C}}$ is a diffeomorphism, we obtain $\pi_{\mathbb{C}}^{-1}(U) \simeq \mathfrak{g} \times G \times U$. The quotient map $\pi : \Phi^{-1}(0) \rightarrow M//G$ is a principal G -bundle, so

there is a slice N in $\Phi^{-1}(0)$ which is transverse to the G -action and diffeomorphic to U . The free G -action on $\Phi^{-1}(0)$ then yields a diffeomorphism from $G \times N$ to a G -invariant neighborhood M_0 in the zero-set. Hence we obtain a local diffeomorphism $\mathfrak{g} \times M_0 \simeq \mathfrak{g} \times G \times N \simeq \pi_{\mathbb{C}}^{-1}(U)$ which corresponds to the map $\Lambda|_{\mathfrak{g} \times M_0}$. This shows, in particular, that the Jacobian of Λ is invertible at each point and hence that Λ is a global diffeomorphism. \square

Proof of Theorem 2.1. Given a point u in $\Phi^{-1}(0)/G$, choose a neighborhood U of u , a point $x \in \Phi^{-1}(0)$ with $\pi(x) = u$, and an embedded submanifold N through x which is transverse to the G -action so that π maps N injectively and diffeomorphically onto U . Let $\pi^{-1} : U \rightarrow N$ be the smooth inverse to π . Now consider the map $\Psi : G_{\mathbb{C}} \times U \rightarrow M_s$ given by

$$\Psi(g, u) = g \cdot \pi^{-1}(u).$$

First, we establish that Ψ is injective (globally, not just locally). If u_1 and u_2 are distinct elements of N , then $\pi^{-1}(u_1)$ and $\pi^{-1}(u_2)$ are in distinct G -orbits. But distinct G -orbits in $\Phi^{-1}(0)$ lie in distinct $G_{\mathbb{C}}$ -orbits (see the comments following the proof of Theorem 4.1(a)). Thus, $\Psi(g_1, u_1) \neq \Psi(g_2, u_2)$ if $u_1 \neq u_2$. Then if $g_1 \neq g_2$, $\Psi(g_1, u) \neq \Psi(g_2, u)$, because $G_{\mathbb{C}}$ acts freely on M_s .

Next, we establish that the differential of Ψ is injective at each point. Since π maps N diffeomorphically onto U , it suffices to prove the same thing for the map $\Gamma : G_{\mathbb{C}} \times N \rightarrow M_s$ given by $\Gamma(g, n) = g \cdot n$. So consider $(g, n) \in G_{\mathbb{C}} \times N$. Since π maps N diffeomorphically onto U , $T_n(\Phi^{-1}(0))$ is the direct sum of $T_n(N)$ and $T_n(G \cdot n)$. Furthermore, for any $\xi \in \mathfrak{g}$, the vector field JX^{ξ} is nonzero at n and orthogonal to the tangent space to $\Phi^{-1}(0)$. It follows that

$$T_n(M) = T_n(G_{\mathbb{C}} \cdot n) \oplus T_n(N)$$

and thus

$$\begin{aligned} T_{gn}(M) &= g_*(T_n(G_{\mathbb{C}} \cdot n)) \oplus g_*(T_n(N)) \\ &= T_{gn}(G_{\mathbb{C}} \cdot n) \oplus g_*(T_n(N)) \end{aligned}$$

since the action of g takes $G_{\mathbb{C}} \cdot n$ into itself. Now suppose that $\alpha(t) = (g(t), n(t))$ is a curve in $G_{\mathbb{C}} \times N$ passing through (g, n) at $t = 0$ and such that $g'(0)g^{-1} = \xi \in \mathfrak{g}_{\mathbb{C}}$ and $n'(t) = X$. Then applying Γ and differentiating at $t = 0$ gives

$$\Gamma_*(\alpha'(0)) = X_{g \cdot n}^{\xi} + g_*(X),$$

which is nonzero provided that $\alpha'(0)$ is nonzero.

Finally, we prove the theorem. Since Ψ is globally injective, it has an inverse map defined on its image, where the image of Ψ is simply $\pi_{\mathbb{C}}^{-1}(U)$. The inverse function theorem tells us, then, that $\pi_{\mathbb{C}}^{-1}(U)$ is an open set and that the inverse map to Ψ is smooth. Thus, $\pi_{\mathbb{C}}^{-1}(U)$ is diffeomorphic to $G_{\mathbb{C}} \times U$ in such a way that the action of $G_{\mathbb{C}}$ corresponds to the left action of $G_{\mathbb{C}}$ on itself. Under this diffeomorphism, $\pi_{\mathbb{C}}$ corresponds to projection onto the second factor, which is smooth. Thus the diffeomorphism Ψ shows that $M_s \rightarrow M//G$ has the necessary smooth local triviality property to be a smooth principal $G_{\mathbb{C}}$ -bundle. \square

2.3 Quantum reduction and the Guillemin–Sternberg map

In the previous section, we considered reduction at the classical level, which amounts to passing from M to $M//G$. Alternatively, we may first quantize M , by looking at the space of holomorphic sections of $l^{\otimes k}$ over M , and then perform reduction at the quantum level. According to the philosophy of Dirac [Dir64, Lecture 2, pp. 34 ff.], reduction at the quantum level amounts to looking at holomorphic sections that are invariant under an appropriate action of the group G , that is, the quantum reduced space is the null-space of the quantized moment map ((2.6) below).

We now describe how the action of G on the space of sections is constructed. Following the program of geometric quantization, we first define an action of the Lie algebra \mathfrak{g} on the space of smooth sections of ℓ by

$$Q_\xi := \nabla_{X\xi} - i\phi_\xi, \quad \xi \in \mathfrak{g}.$$

These operators satisfy $[Q_\xi, Q_\eta] = Q_{[\xi, \eta]}$. The prequantum bundle ℓ is said to be G -invariant if this action of \mathfrak{g} can be exponentiated to an action of the group G . (If G is simply connected, this can always be done.) We henceforth assume that ℓ is G -invariant. Since ℓ is a holomorphic line bundle (i.e., the total space is a complex manifold), it is not hard to show that the G -action on ℓ can be analytically continued to an action of $G_{\mathbb{C}}$ (by following an argument similar to that of Guillemin and Sternberg that the action of G on M can be analytically continued to an action of $G_{\mathbb{C}}$ on M [GS82, Theorem 4.4]).

For each k , we then define an action of \mathfrak{g} on the space of smooth sections of $\ell^{\otimes k}$ by

$$Q_\xi := \nabla_{X\xi} - ik\phi_\xi, \quad \xi \in \mathfrak{g}. \quad (2.6)$$

(We suppress the dependence on k .) These also satisfy $[Q_\xi, Q_\eta] = Q_{[\xi, \eta]}$ and they preserve the quantum Hilbert space $\mathcal{H}(M; \ell^{\otimes k})$ of *holomorphic* sections of $\ell^{\otimes k}$. Since ℓ is G -invariant, it follows that these operators can be exponentiated to an action of G that preserves $\mathcal{H}(M; \ell^{\otimes k})$. The space obtained by performing reduction at the quantum level is then the space of G -invariant holomorphic sections of $\ell^{\otimes k}$, denoted $\mathcal{H}(M; \ell^{\otimes k})^G$.

We see, then, that if we first quantize M and then reduce by G , we obtain the space $\mathcal{H}(M; \ell^{\otimes k})^G$ of G -invariant sections of $\ell^{\otimes k}$ over M . On the other hand, we may first reduce M by G at the classical level and then perform quantization of the reduced manifold $M//G$. Assuming that the bundle ℓ is G -invariant, it is not hard to see that it descends naturally to a Hermitian line bundle $\hat{\ell}$ with connection over $M//G$, whose curvature is equal to $-i\hat{\omega}$. The bundle $\hat{\ell}$ can be made into a holomorphic line bundle in the same way as ℓ , by decreeing that the holomorphic sections are those that are covariantly constant in the $(0, 1)$ -directions. The space, then, obtained by first reducing and then quantizing is the space of all holomorphic sections of $\hat{\ell}^{\otimes k}$ over $M//G$, denoted $\mathcal{H}(M//G; \hat{\ell}^{\otimes k})$.

In the paper [GS82], Guillemin and Sternberg consider a geometrically natural linear map A_k between the “first quantize and then reduce” space $\mathcal{H}(M; \ell^{\otimes k})^G$ and the “first reduce and then quantize” space $\mathcal{H}(M//G; \hat{\ell}^{\otimes k})$. This map consists simply of taking a G -invariant holomorphic section over M , restricting it to $\Phi^{-1}(0)$, and then letting it descend to $M//G = \Phi^{-1}(0)/G$. The remarkable result established in [GS82, Thm 5.2] is that this natural map is invertible. This result has been generalized in various ways, to symplectic manifolds that are not Kähler and to situations where the quotient is singular. See, for example, [Hue06], [JK97], [Sja95], [Sja96], [Mei98], and [TZ98].

In view of the importance of the map A_k , we briefly sketch the proof of its invertibility. It is easy to see that a section of $\hat{\ell}$ defines a G -invariant section of ℓ over the zero-set by pullback along the canonical projection $\pi : \Phi^{-1}(0) \rightarrow M//G$, and vice-versa. Moreover, a G -invariant section over the zero-set has a unique analytic continuation to the stable set, obtained from the $G_{\mathbb{C}}$ -action on M . The difficulty is in showing that the resulting analytic continuation extends smoothly across the unstable set. To construct this extension, Guillemin and Sternberg show that the magnitude of a G -invariant holomorphic section over the stable set approaches zero as one approaches the unstable set. The Riemann Extension Theorem then shows that such a section extends holomorphically across the unstable set.

The invertibility of the map A_k shows that, *in a certain sense*, quantization commutes with reduction. That is, A_k gives a natural identification of the *vector space* obtained by first quantizing and then reducing with the *vector space* obtained by first reducing and then quantizing. However, in quantum mechanics, the structure of the spaces as *Hilbert spaces*, not just vector spaces, is essential. For example, the expectation value of an operator D in the “state” s is given by $\langle s, Ds \rangle$,

where s is a unit vector in the relevant Hilbert space. It is natural, then, to consider the extent to which the map A_k is unitary with respect to the natural inner products (1.2) and (1.3) on the two spaces.

Our Theorem 4.2 suggests that it is very rare for A_k (or any constant multiple of A_k) to be unitary. If the G -orbits in $\Phi^{-1}(0)$ have nonconstant volume, Theorem 5.3 shows that for all sufficiently large k , A_k is not a constant multiple of a unitary map. Indeed, in such cases, A_k is not even asymptotic to a multiple of a unitary map in the limit as Planck's constant tends to zero (i.e., as k tends to infinity). We may say therefore that, in the setting of [GS82], quantization does *not* commute with reduction in the strongest desirable sense.

3 A map of Guillemin–Sternberg type in the presence of the metaplectic correction

In this section, we consider the “metaplectic correction,” which consists of tensoring the original line bundle ℓ by the square root of the canonical bundle of M , and similarly for $\hat{\ell}$. We introduce here an analog B_k of the Guillemin–Sternberg map in the presence of the metaplectic correction and we show (Theorem 3.2 of this section) that B_k is invertible for all sufficiently large k . We will see eventually (Theorems 5.2 and 5.3) that the maps B_k , unlike the maps A_k , become approximately unitary as Planck's constant tends to zero (i.e., as k tends to infinity).

There are two conceptual differences between the original map A_k and the “metaplectically corrected” map B_k . First, G -invariant holomorphic sections of the square root of the canonical bundle behave badly as one approaches the unstable set. To compensate for this, we need rapid decay of holomorphic sections of G -invariant holomorphic sections of $\ell^{\otimes k}$. This means that we obtain invertibility of the maps B_k for all sufficiently large k , rather than for all k , as is the case for the maps A_k . Second, the pointwise magnitude of a section $B_k s$ does not agree with the pointwise magnitude of the original section s on $\Phi^{-1}(0)$, in contrast to the map A_k . Rather, the pointwise magnitudes differ by a factor involving the volume of the G -orbits in $\Phi^{-1}(0)$ (Theorem 3.3). The volume factor in Theorem 3.3 ultimately cancels an unwanted volume factor in the asymptotics of the maps A_k (Section 5), allowing the maps B_k to be asymptotically unitary even though maps A_k are not.

3.1 Half-form bundles on M and $M//G$

Let $K = \bigwedge^n (T^{1,0}M)^*$ denote the canonical bundle of M , that is, the top exterior power of the bundle of $(1,0)$ -forms. A smooth section of K is called an $(n,0)$ -form, and the set of such forms is denoted $\Omega^{n,0}(M)$. An $(n,0)$ -form is called holomorphic if in each holomorphic local coordinate system, the coefficient of $dz_1 \wedge \cdots \wedge dz_n$ is a holomorphic function. Equivalently, we may define a “partial connection” (defined only for vector fields of type $(0,1)$) on the space of $(n,0)$ -forms by setting

$$\nabla_X \alpha = i_X(d\alpha)$$

whenever X is of type $(0,1)$. It is easily verified that α is holomorphic if and only if $\nabla_X \alpha = 0$ for all vector fields of type $(0,1)$. (Compare [Woo91, Sect. 9.3].)

A choice of square root \sqrt{K} of the canonical bundle, if it exists, is called a half-form bundle. Since the first Chern class of the canonical bundle is $-c_1(M)$, the canonical bundle will admit a square root if and only if $-c_1(M)/2$ is an integral class; that is, if and only if the second Stiefel–Whitney class $w_2(M)$ (which is the reduction mod 2 of the first Chern class for a complex manifold) vanishes. We now assume that $-c_1(M)/2$ is integral and we fix a choice of \sqrt{K} . (It is likely that results similar to the ones in this paper could be obtained assuming that $[\omega/2\pi] - c_1(M)/2$ is integral, rather than assuming, as we do, that $[\omega/2\pi]$ and $c_1(M)/2$ are separately integral. See [Czy78] or [Woo91, Sect. 10.4].)

One can define a partial connection acting on sections ν of \sqrt{K} by requiring that

$$2(\nabla_X \nu) \nu = \nabla_X(\nu^2). \quad (3.1)$$

(See again [Woo91, Sect. 9.3].) The bundle \sqrt{K} can then be made into a holomorphic line bundle by defining the holomorphic sections of \sqrt{K} to be those for which $\nabla_X \nu = 0$ for all vector fields of type $(0, 1)$.

Because the action of G on M is holomorphic, the action of G on n -forms preserves the space of $(n, 0)$ -forms and the space of holomorphic n -forms. The associated action of a Lie algebra element $\xi \in \mathfrak{g}$ on $(n, 0)$ -forms is by the Lie derivative $\mathcal{L}_{X\xi}$. There is an associated action of \mathfrak{g} on the space of sections of \sqrt{K} , also denoted $\mathcal{L}_{X\xi}$, satisfying (by analogy with (3.1))

$$2(\mathcal{L}_{X\xi} \nu) \nu = \mathcal{L}_{X\xi}(\nu^2), \quad (3.2)$$

and this action preserves the space of holomorphic sections.

There is a natural way to define a Hermitian structure on \sqrt{K} , which is standard in geometric quantization. (It is a special case of the BKS pairing; see [Woo91, Sec. 10.4].) If $\nu, \mu \in \Gamma(\sqrt{K})$ are half-forms, then $\nu^2 \wedge \bar{\mu}^2 \in \Gamma(\bigwedge^{2n} T_{\mathbb{C}}^*(M))$. The volume form $\varepsilon_\omega = \omega^{\wedge n}/n!$ is a global trivializing section of the determinant bundle $\bigwedge^{2n} T^*M$, and so there is a function, denoted by (ν, μ) , such that

$$\nu^2 \wedge \bar{\mu}^2 = (\nu, \mu)^2 \varepsilon_\omega. \quad (3.3)$$

We can use this pairing to define a Hermitian form on the tensor product $\ell^{\otimes k} \otimes \sqrt{K}$: for sections $t_1, t_2 \in \Gamma(\ell^{\otimes k} \otimes \sqrt{K})$ which are locally represented by $t_j(x) = s_j(x)\mu(x)$, we define

$$(t_1, t_2)(x) = (s_1\nu, s_2\mu)(x) = (s_1(x), s_2(x))(\nu, \mu)(x).$$

Denote the pairing of a half-form with itself by $|\nu|^2 = (\nu, \nu)$.

Let \widehat{K} denote the canonical bundle over the reduced manifold $M//G$. We now assume that the action (3.2) of \mathfrak{g} on sections of \sqrt{K} exponentiates to an action of the group G (this is automatic, for example, if G is simply connected). In the next subsection, we will show that this assumption allows us to construct in a natural way a square root $\sqrt{\widehat{K}}$ of \widehat{K} that is related in a nice way to the chosen square root \sqrt{K} of K . Once $\sqrt{\widehat{K}}$ is constructed, we will define a Hermitian structure on $\sqrt{\widehat{K}}$ analogously to (3.3):

$$\nu^2 \wedge \bar{\mu}^2 = (\nu, \mu)^2 \varepsilon_{\widehat{\omega}}, \quad (3.4)$$

where $\varepsilon_{\widehat{\omega}} = \widehat{\omega}^{\wedge(n-d)}/(n-d)!$ is the volume form on $M//G$.

3.2 The modified Guillemin–Sternberg map

We continue to assume that the canonical bundle K of M admits a square root and that we have chosen one such square root and denoted it by \sqrt{K} . We also continue to assume that the action of the Lie algebra \mathfrak{g} on sections of \sqrt{K} , given by (3.2), exponentiates to an action of the group G .

In this subsection, we describe a natural map B_k from the space of G -invariant holomorphic sections of $\ell^{\otimes k} \otimes \sqrt{K}$ to the space of holomorphic sections of $\widehat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}}$. This map is the analog of the Guillemin–Sternberg map A_k between the G -invariant holomorphic sections of $\ell^{\otimes k}$ and the space of holomorphic sections of $\widehat{\ell}^{\otimes k}$. However, the construction of B_k is more involved than that of A_k . After all, sections of \sqrt{K} are square roots of $(n, 0)$ -forms, whereas sections of $\sqrt{\widehat{K}}$ are square roots of $(n-d, 0)$ -forms. Thus the map B_k must include a mechanism for changing the degree of a half-form.

There is one further, vitally important difference between the map B_k and the map A_k . The bundle $\widehat{\ell}$ inherits its Hermitian structure from that of ℓ . As a result, the pointwise magnitude of

a section a G -invariant section of $\ell^{\otimes k}$ is the same on $\Phi^{-1}(0)$ as the pointwise magnitude of the corresponding section of $\hat{\ell}^{\otimes k}$. That is, for each $s \in \mathcal{H}(M; \ell^{\otimes k})^G$ and each $x_0 \in \Phi^{-1}(0)$ we have

$$|s|^2(x_0) = |A_k s|^2([x_0]). \quad (3.5)$$

By contrast, $\sqrt{\widehat{K}}$ has its own intrinsically defined Hermitian structure given by (3.3). It turns out, then, that B_k does not satisfy the analog of (3.5). Rather, we will show (Theorem 3.3) that for $r \in \mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{\widehat{K}})^G$ and each $x_0 \in \Phi^{-1}(0)$ we have

$$|r|^2(x_0) = 2^{d/2} \text{vol}(G \cdot x_0)^{-1} |B_k r|^2([x_0]). \quad (3.6)$$

We will see eventually that the volume factor in (3.6) cancels an unwanted volume factor in the asymptotic behavior of A_k . The result is that maps B_k (unlike the maps A_k) are asymptotically unitary as k tends to infinity.

Finally, we address the invertibility of the map B_k . Again, the situation is slightly different from that with A_k . Half-forms typically have bad behavior near the unstable set, and as a result, we are only able to prove that B_k is invertible for *sufficiently large* k . (This is in contrast to the map A_k , which is invertible for all k .)

Before we get to half-forms, it will be useful to describe the descent of $G_{\mathbb{C}}$ -invariant $(n, 0)$ -forms on the stable set M_s to $(n - d, 0)$ -forms on $M//G$. We cannot simply restrict an $(n, 0)$ -form to the zero-set and then let it descend to $M//G$, as we did for the bundle $\ell^{\otimes k}$, because the result would not be an $(n - d)$ -form on $M//G$. The process, which we describe in detail below, is to first contract with the infinitesimal G -directions and then use $\pi_{\mathbb{C}}$ to push the result down to the quotient. It turns out that this process is invertible and preserves holomorphicity.

Choose an Ad-invariant inner product on \mathfrak{g} which is normalized so that the volume of G with respect to the associated Haar measure is 1. Then fix a basis $\Xi = \{\xi_1, \dots, \xi_d\}$ of the Lie algebra \mathfrak{g} which is orthonormal with respect to this inner product. Given a $G_{\mathbb{C}}$ -invariant $(n, 0)$ -form defined on the stable set M_s , define an $(n - d, 0)$ -form β on M_s by

$$\beta = i(\bigwedge_j X^{\xi_j})\alpha.$$

We claim that β has the properties that for each $X \in T(G_{\mathbb{C}} \cdot x)$

$$i(X)\beta = 0 \text{ and} \quad (3.7)$$

$$i(X)d\beta = 0. \quad (3.8)$$

It is clear that (3.7) holds when X is of the form X^{ξ} with $\xi \in \mathfrak{g}$. But since β is an $(n - d, 0)$ -form, $i(JX^{\xi})\beta = \sqrt{-1}i(X^{\xi})\beta$, and so (3.7) holds for $X = X^{\xi}$ with $\xi \in \mathfrak{g}_{\mathbb{C}}$. To verify (3.8), we first note that in the presence of (3.7), (3.8) is equivalent to $\mathcal{L}_X(\beta) = 0$, i.e., to the condition that β be $G_{\mathbb{C}}$ -invariant. So we need to verify that β is invariant if α is. Since α is of type $(n, 0)$, contracting α with $\bigwedge_j X^{\xi_j}$ is the same as contracting it with $\bigwedge_j \pi_+ X^{\xi_j}$, and a simple computation shows that the polyvector $\bigwedge_j \pi_+ X^{\xi_j}$ is $G_{\mathbb{C}}$ -invariant. Contracting a $G_{\mathbb{C}}$ -invariant form with a $G_{\mathbb{C}}$ -invariant polyvector gives another $G_{\mathbb{C}}$ -invariant form.

Now, given any $(n - d, 0)$ -form β on M_s satisfying (3.7) and (3.8), it is not hard to show that there is a unique $(n - d, 0)$ -form $\hat{\beta}$ on $M//G = M_s/G_{\mathbb{C}}$ such that $\pi_{\mathbb{C}}^*(\hat{\beta}) = \beta$, where $\pi_{\mathbb{C}}^* : M_s \rightarrow M_s/G_{\mathbb{C}}$ is the quotient map. So we have given a procedure for turning a $G_{\mathbb{C}}$ -invariant $(n, 0)$ -form α on M into an $(n - d, 0)$ -form $\hat{\beta}$ on $M//G$.

In the other direction, suppose $\hat{\beta}$ is an $(n - d, 0)$ -form on $M//G$. Then the pullback $\beta := \pi_{\mathbb{C}}^*(\hat{\beta})$ is an $(n - d, 0)$ -form on M_s that (as is easily verified) satisfies (3.7) and (3.8). We can construct from β an $(n, 0)$ -form α as follows: given a local frame $\{X^{\xi_1}, \dots, X^{\xi_d}, Y_1, \dots, Y_{n-d}\}$ for $T_x M_s$, set

$$\alpha(X^{\xi_1}, \dots, X^{\xi_d}, Y_1, \dots, Y_{n-d}) = \pi_{\mathbb{C}}^*\hat{\beta}(Y_1, \dots, Y_{n-d}) \quad (3.9)$$

and define α on any other frame by $GL(n, \mathbb{C})$ -equivariance and the requirement that α be an $(n, 0)$ -form. (Note that the tangent space at a point in the stable set is a direct sum of the tangent space to the $G_{\mathbb{C}}$ -orbit through that point and the transverse directions, so that every frame is equivalent to a linear combination of frames which are $GL(n, \mathbb{C})$ -equivalent to one of the form $(W_1, \dots, W_d, Y_1, \dots, Y_{n-d})$ where $W_j = X^{\xi_j}$ or JX^{ξ_j} ; we define $i(JX^{\xi})\alpha = \sqrt{-1}i(X^{\xi})\alpha$.) It is easily verified that α is again $G_{\mathbb{C}}$ -invariant.

The two processes we have defined—contracting a $G_{\mathbb{C}}$ -invariant $(n, 0)$ -form on M_s with $\bigwedge_j X^{\xi_j}$ and the letting it descent to the quotient, and pulling back an $(n-d, 0)$ -form on $M//G$ and then “expanding” it by (3.9)—are clearly inverse to each other. They therefore define a bijective map

$$\alpha \in \Omega^{n,0}(M_s)^{G_{\mathbb{C}}} \mapsto \mathfrak{B}(\alpha) \in \Omega^{n-d,0}(M//G) \quad (3.10)$$

where $\mathfrak{B}(\alpha)$ is the unique $(n-d, 0)$ -form on $M//G$ such that $\pi_{\mathbb{C}}^* \mathfrak{B}(\alpha) = i(\bigwedge_j X^{\xi_j})\alpha$. This bijective map has the further property that α is locally holomorphic if and only if $\mathfrak{B}(\alpha)$ is locally holomorphic since contracting with $\bigwedge_j X^{\xi_j}$ is the same as contracting with $\bigwedge_j \pi_+ X^{\xi_j}$ and the vector fields $\pi_+ X^{\xi}$ are holomorphic.

Finally, note that the contraction $i(X^{\xi_j})\alpha$ is proportional to the contraction $i(JX^{\xi_j})\alpha$. By the results of Section 2.2, pushing down a $G_{\mathbb{C}}$ -invariant $(n, 0)$ -form by the above process is equivalent to first restricting it to $\Phi^{-1}(0)$ and then pushing it down by the analogous process using the quotient map $\pi : \Phi^{-1}(0) \rightarrow M//G \simeq \Phi^{-1}(0)/G$. The vectors JX^{ξ} span the normal bundle of $\Phi^{-1}(0)$ in M , so the contraction step can then be understood as contracting with the directions normal to the zero-set; this is perhaps the obvious way to “restrict” a top-dimensional form to a submanifold. Since the Guillemin–Sternberg map A_k is defined as “restrict to $\Phi^{-1}(0)$ and then descend to $M//G$ ”, we will sometimes interpret the map \mathfrak{B} as first contracting, then restricting the result to $\Phi^{-1}(0)$, and finally pushing the result to the quotient.

We now turn to the descent map for half-forms. Recall that we assume that the infinitesimal action of \mathfrak{g} on \sqrt{K} defined by (3.2) exponentiates to an action of G on \sqrt{K} which is compatible with the action of G on K . Following essentially the same argument that Guillemin and Sternberg make for the analytic continuation of the action of G on M , it can be shown that this action lifts to an action of $G_{\mathbb{C}}$ which covers the $G_{\mathbb{C}}$ -action on K .

A moment’s thought shows that the map \mathfrak{B} actually provides an identification of K_x with $\widehat{K}_{[x]}$, $x \in M_s$, and this identification commutes with the action of $G_{\mathbb{C}}$, that is, for each $g \in G_{\mathbb{C}}$

$$g^* \mathfrak{B}(\alpha) = \mathfrak{B}(g^* \alpha).$$

The contraction map \mathfrak{B} therefore identifies $K|_{G_{\mathbb{C}} \cdot x}$ with $\widehat{K}_{[x]}$.

Let $\sqrt{\widehat{K}}$ denote a line bundle over $M//G$ whose fiber is the equivalence class of $\sqrt{K}|_{G_{\mathbb{C}} \cdot x}$ under the $G_{\mathbb{C}}$ -action. Since tensoring commutes with the $G_{\mathbb{C}}$ -action, it follows that this bundle is a square root of the canonical bundle on $M//G$. We let $\Gamma(M, \sqrt{K})$ and $\Gamma(M//G, \sqrt{\widehat{K}})$ denote the space of smooth sections of \sqrt{K} and $\sqrt{\widehat{K}}$, respectively, and we let $\Gamma(M, \sqrt{K})^G$ denote the space of G -invariant sections of \sqrt{K} .

By construction, then, the pullback of the half-form bundle on the quotient by the quotient map is isomorphic to the half-form bundle on the stable set; i.e.,

$$\pi_{\mathbb{C}}^* \sqrt{\widehat{K}} \simeq \sqrt{K}|_{M_s}.$$

Moreover, this isomorphism defines a map

$$B : \Gamma(M, \sqrt{K})^{G_{\mathbb{C}}} \rightarrow \Gamma(M//G, \sqrt{\widehat{K}})$$

such that $(B\nu_x, B\nu_x) = \mathfrak{B}(\nu^2)$.

We have thus proved the following.

Theorem 3.1. *There exists a linear map $B : \Gamma(M, \sqrt{K})^G \rightarrow \Gamma(M//G, \sqrt{\widehat{K}})$, unique up to an overall sign, with the property that*

$$\pi_{\mathbb{C}}^*[(B\nu)^2] = \left[i \left(\bigwedge_j X^{\xi_j} \right) (\nu^2) \right] \Big|_{M_s}.$$

For any open set U in $M//G$, if ν is holomorphic in a neighborhood V of $\pi_{\mathbb{C}}^{-1}(U)$, then $B\nu$ is holomorphic on U .

For each k , there is a linear map $B_k : \Gamma(M, \ell^{\otimes k} \otimes \sqrt{K})^G \rightarrow \Gamma(M//G, \hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}})$, unique up to an overall sign, with the property that

$$B_k(s \otimes \nu) = A_k(s) \otimes B(\nu)$$

for all $s \in \Gamma(\ell^{\otimes k})$ and $\nu \in \Gamma(\sqrt{K})$. This map takes holomorphic sections of $\ell^{\otimes k} \otimes \sqrt{K} \Big|_V$ to holomorphic sections of $\hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}} \Big|_U$.

The last claim in Theorem 3.1 follows from the definition (3.1) of the partial connection on \sqrt{K} .

We obtain a version of Guillemin and Sternberg's "quantization commutes with reduction" using the modified map B_k . Our version is necessarily weaker since the pointwise behavior of G -invariant elements of $\mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{K})$ is worse than the behavior of G -invariant sections of $\mathcal{H}(M; \ell^{\otimes k})$ (Theorem 4.1).

Theorem 3.2. *For k sufficiently large, the map*

$$B_k : \mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{K})^G \rightarrow \mathcal{H}(M//G; \hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}})$$

is bijective.

The proof of this result (below) is similar to the proof in [GS82] of the invertibility of the map A_k , with a few modifications to deal with the half-forms.

The inclusion of half-forms modifies the Hermitian form on sections of the reduced corrected prequantum bundle $\hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}}$ in a way that is not proportional to the modification of Hermitian form on sections of $\ell^{\otimes k} \otimes \sqrt{K}$. We can actually compute the difference:

Theorem 3.3. *Suppose $r \in \mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{K})$. Then for $x_0 \in \Phi^{-1}(0)$*

$$|B_k r|^2([x_0]) = 2^{-d/2} \text{vol}(G \cdot x_0) |r|^2(x_0).$$

The factor of $2^{-d/2} \text{vol}(G \cdot x_0)$ appearing in Theorem 3.3 is the ultimate reason that the modified Guillemin–Sternberg map is asymptotically unitary; as we will see in Section 5, it precisely cancels the leading order asymptotic value of the uncorrected density I_k .

Proof of Theorem 3.2. The natural map is injective because two holomorphic sections which agree on the stable set M_s , which is an open dense subset of M , must necessarily be equal. We now establish surjectivity for large k . A section $\hat{r} \in \mathcal{H}(M//G; \hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}})$ determines a G -invariant section $\pi^* \hat{r}$ over the zero-set $\Phi^{-1}(0)$. Using the action of $G_{\mathbb{C}}$ on the bundles, we can extend $\pi^* \hat{r}$ uniquely to a holomorphic, $G_{\mathbb{C}}$ -invariant section of $\ell^{\otimes k} \otimes \sqrt{K}$ defined over the stable set M_s . As shown in [GS82, Appendix], there exists some k for which $\mathcal{H}(M; \ell^{\otimes k})$ contains some nonzero G -invariant section s . By (2.5), the unstable set is contained in the zero-set of s . Thus, if the magnitude of r remains bounded as we approach the unstable set, the Riemann Extension Theorem (e.g., [GH78, pp. 9]) will imply that r extends holomorphically to all of M . Suppose

$\xi \in \mathfrak{g}$ has $|\xi| = 1$. Then we will show in the next section (Theorem 4.1) that the variation of the magnitude $|r|^2$ along the curve $e^{it\xi} \cdot x_0$ is

$$\frac{d}{dt} |r|^2(e^{it\xi} \cdot x_0) = |r|^2(x_0) \left[-2k \phi_\xi(e^{it\xi} \cdot x_0) - \frac{\mathfrak{L}_{JX\xi\varepsilon_\omega}}{2\varepsilon_\omega}(e^{it\xi} \cdot x_0) \right]$$

where $\varepsilon_\omega = \omega^{\wedge n}/n!$ is the Liouville volume form. Furthermore, we will show in (4.6) that $\phi_\xi(e^{it\xi} \cdot x_0)$ increases with t for $t \geq 0$. As a result, the function $\phi_\xi(e^{i\xi} \cdot x_0)$, as ξ varies over unit vectors in \mathfrak{g} and x_0 varies over $\Phi^{-1}(0)$, is strictly positive and thus bounded below by compactness. Meanwhile, $\mathfrak{L}_{JX\xi\varepsilon_\omega}/\varepsilon_\omega$ is bounded over M uniformly in ξ with $|\xi| = 1$, again by compactness. Using the monotonicity of $\phi_\xi(e^{it\xi} \cdot x_0)$ in t , it follows that for all sufficiently large k ,

$$2k\phi_\xi(e^{it\xi} \cdot x_0) \geq \left| \frac{\mathfrak{L}_{JX\xi\varepsilon_\omega}}{2\varepsilon_\omega}(e^{it\xi} \cdot x_0) \right|$$

for all $x_0 \in \Phi^{-1}(0)$, all $\xi \in \mathfrak{g}$ with $|\xi| = 1$, and all $t \geq 1$. It follows that if k is large enough, every r obtained in the above way will extend holomorphically to all of M . \square

The proof of Theorem 3.3 boils down to the following two lemmas. Recall that $\Xi = \{\xi_1, \dots, \xi_d\}$ is a basis of \mathfrak{g} for which the associated Haar measure on G is normalized to 1.

Lemma 3.4. *The function $\sqrt{\det_\Xi B_x} := \det(B(X^{\xi_j}, X^{\xi_k}))$ is constant along each G -orbit. Moreover,*

$$\sqrt{\det_\Xi B_x} = \text{vol}(G \cdot x).$$

Proof. The fixed basis Ξ of \mathfrak{g} defines a left-invariant coframe ϑ on $G \cdot x$. With respect to this coframe, the Riemannian volume associated to B is

$$d\text{vol} = \sqrt{\det_\Xi B} \vartheta^1 \wedge \dots \wedge \vartheta^k.$$

By our definition of Ξ , the pullback $dg_0 = \varphi^* \vartheta^1 \wedge \dots \wedge \vartheta^k$ to G by the map $\varphi : G \rightarrow G \cdot x$ is a left Haar measure on G for which the corresponding volume is $\text{vol}_0(G) = 1$. Denote by $d\text{vol}_0(G) = \vartheta^1 \wedge \dots \wedge \vartheta^k$. Then since our choice of inner product on \mathfrak{g} yields $\int_{G \cdot x} d\text{vol}_0(G) = 1$, integrating the equation $d\text{vol} = \sqrt{\det_\Xi B} d\text{vol}_0(G)$ over $G \cdot x$ yields the desired result. \square

Observe that if we choose a basis Ξ which is not orthonormal with respect to our fixed normalized inner product on G , or if we choose an inner product on \mathfrak{g} that is not normalized so that $\int_G dg = 1$, then Lemma 3.4 yields $\sqrt{\det_\Xi B_x} = C \text{vol}(G \cdot x)$. The final effect of our seemingly natural choices (which make C equal 1) is that the density J_k (which takes into account the metaplectic correction) approaches 1 as $k \rightarrow \infty$ (in general it would be C^{-1}).

Our second lemma is technical; it is a straightforward though somewhat tedious calculation.

Lemma 3.5. *Let $Z^j = \pi_+ X^{\xi_j}$. Then for $x_0 \in \Phi^{-1}(0)$ we have*

$$\mathbf{i}(\bigwedge_j Z^j) \circ \mathbf{i}(\bigwedge_k \bar{Z}^j) \varepsilon_\omega(x_0) \Big|_{\Phi^{-1}(0)} = 2^{-d} (\text{vol}(G \cdot x_0))^2 \frac{\omega^{n-d}}{(n-d)!}(x_0) \Big|_{\Phi^{-1}(0)}.$$

Proof. First, since M is Kähler, the symplectic form ω is of (1,1)-type, that is, ω contracted on two holomorphic or two antiholomorphic vectors is zero. With our convention that $\bigwedge_j w^j := \frac{1}{d!} \sum_{\sigma \in S_d} w^{\sigma(1)} \otimes \dots \otimes w^{\sigma(d)}$ we get $\mathbf{i}(\bigwedge_j w^j) \alpha = \alpha(w^1, \dots, w^d, \cdot, \dots, \cdot) = \mathbf{i}(w^d) \circ \dots \circ \mathbf{i}(w^1) \alpha$. Using the fact that the interior product is an antiderivation, we compute

$$\mathbf{i}(\bar{Z}^d) \circ \dots \circ \mathbf{i}(\bar{Z}^1) \omega^n = (-1)^{d(d-1)/2+d} \frac{n!}{(n-d)!} \bigwedge_j \omega(\cdot, \bar{Z}^j) \wedge \omega^{n-d}.$$

Continuing on, we contract the above with the holomorphic polyvector:

$$\begin{aligned} \mathbf{i}(\bigwedge_j Z^j) \circ \mathbf{i}(\bigwedge_k \bar{Z}^k) \omega^n &= (-1)^{d(d+1)/2} \frac{n!}{(n-d)!} [\det(\omega(Z^j, \bar{Z}^k)) \omega^{n-d} \\ &\quad + (-1)^d \bigwedge_j \omega(\cdot, \bar{Z}^j) \wedge (n-d)! \bigwedge_k \omega(Z^k, \cdot) \wedge \omega^{n-2d}]. \end{aligned} \quad (3.11)$$

A short computation shows that $\omega(Z^j, \bar{Z}^k) = \frac{i}{2} B(X^{\xi_j}, X^{\xi_k})$. When restricted to $\Phi^{-1}(0)$, the moment map is constant. Hence

$$\omega(\cdot, \pi_{\pm} X^{\xi_j})|_{\Phi^{-1}(0)} = \frac{1}{2} d\phi_{\xi_j} \mp \frac{i}{2} \omega(\cdot, JX^{\xi_j})|_{\Phi^{-1}(0)} = \mp \frac{i}{2} \omega(\cdot, JX^{\xi_j})|_{\Phi^{-1}(0)}$$

and so $\bigwedge_j \omega(\cdot, \bar{Z}^j) \wedge \bigwedge_k \omega(Z^k, \cdot)|_{\Phi^{-1}(0)} = 0$. Equation (3.11) is now

$$\mathbf{i}(\bigwedge_j Z^j) \circ \mathbf{i}(\bigwedge_k \bar{Z}^k) \frac{\omega^n}{n!} \Big|_{\Phi^{-1}(0)} = 2^{-d} (\det \Xi B) \frac{\omega^{n-d}}{(n-d)!} \Big|_{\Phi^{-1}(0)}.$$

Combining this with Lemma 3.4 yields the desired result. \square

Proof of Theorem 3.3. Near x_0 we can write $r = s\nu$ where s is a local G -invariant holomorphic section of $\ell^{\otimes k}$ and ν is a local G -invariant holomorphic section of \sqrt{K} . Let $\alpha = \nu^2$. Then since $\mathbf{i}(\pi_+ X^{\xi}) \bar{\alpha} = \mathbf{i}(\pi_- X^{\xi}) \alpha = 0$, we have

$$\begin{aligned} \pi^* (|B\nu|^2 \varepsilon_{\hat{\omega}}) &= (\pi^* B\nu, \pi^* B\nu)^2 \frac{\omega^{n-d}}{(n-d)!} \Big|_{\Phi^{-1}(0)} \\ &= \mathbf{i}(\bigwedge_j Z^j) \alpha \wedge \mathbf{i}(\bigwedge_k \bar{Z}^k) \bar{\alpha} \Big|_{\Phi^{-1}(0)} \\ &= \mathbf{i}(\bigwedge_j Z^j \wedge \bigwedge_k \bar{Z}^k) ((\nu, \nu)^2 \frac{\omega^n}{n!}) \Big|_{\Phi^{-1}(0)} \\ &= (\nu, \nu)^2 2^{-d} \det \Xi B \frac{\omega^{n-d}}{(n-d)!} \Big|_{\Phi^{-1}(0)} \end{aligned}$$

The result now follows upon dividing by $\omega^{n-d}/(n-d)!$, taking the square root and using Lemma 3.4 and the fact that $|A_k s|^2([x_0]) = |s|^2(x_0)$. \square

4 Norm decompositions

In this section, we show how the norm-squared of a G -invariant holomorphic section over the *upstairs* manifold M can be computed as an integral over the *downstairs* manifold $M//G$. To do this, we first show that the pointwise magnitude of a G -invariant holomorphic section (with or without the metaplectic correction) varies in a predictable way as we move off the zero-set of the moment map along the orbits of $G_{\mathbb{C}}$. This means that the behavior of an invariant section on $\Phi^{-1}(0)$ determines its behavior on the whole stable set M_s . We then integrate the resulting expressions over the stable set using the decomposition in Theorem 2.3. By (2.5), the unstable set is contained in a set of complex codimension at least 1 and hence the stable set is a set of full measure in M . A similar computation shows that a Toeplitz operator on the upstairs manifold with a G -invariant symbol, when restricted to the space of G -invariant sections, is equivalent to a certain Toeplitz on the downstairs manifold.

Recall from Section 2.2 that the stable set M_s consists of those points in M that can be moved into the zero-set of the moment map by means of the action of $G_{\mathbb{C}}$. Recall also (Theorem 2.3) that every point in M_s can be expressed uniquely in the form $e^{i\xi} \cdot x_0$, for some $\xi \in \mathfrak{g}$ and $x_0 \in \Phi^{-1}(0)$. The first main result of this section shows how the (pointwise) magnitude of a G -invariant holomorphic section, evaluated at $e^{i\xi} \cdot x_0$, varies with respect to ξ .

Theorem 4.1. *Let s be a G -invariant holomorphic section of $\ell^{\otimes k}$ and let r be a G -invariant holomorphic section of $\ell^{\otimes k} \otimes \sqrt{K}$. Let x_0 be a point in $\Phi^{-1}(0)$ and ξ an element of \mathfrak{g} . Then the magnitudes of the sections at $e^{i\xi} \cdot x_0$ are related to the magnitudes at x_0 as follows:*

$$(a) |s|^2(e^{i\xi} \cdot x_0) = |s|^2(x_0) \exp \left\{ - \int_0^1 2k\phi_\xi(e^{it\xi} \cdot x_0) dt \right\} \quad (4.1)$$

$$(b) |r|^2(e^{i\xi} \cdot x_0) = |r|^2(x_0) \exp \left\{ - \int_0^1 \left(2k\phi_\xi(e^{it\xi} \cdot x_0) + \frac{\mathcal{L}_{JX^\xi} \varepsilon_\omega}{2\varepsilon_\omega}(e^{it\xi} \cdot x_0) \right) dt \right\} \quad (4.2)$$

Furthermore, if we consider

$$\rho(\xi, x_0) := 2 \int_0^1 \phi_\xi(e^{it\xi} \cdot x_0) dt,$$

then for each fixed x_0 , $\rho(\xi, x_0)$ achieves its unique minimum at $\xi = 0$ and the Hessian of $\rho(\xi, x_0)$ at $\xi = 0$ is given by

$$D_{\xi_1} D_{\xi_2} \rho(\xi, x_0)|_{\xi=0} = 2B_{x_0}(JX^{\xi_1}, JX^{\xi_2}), \quad \xi_1, \xi_2 \in \mathfrak{g}.$$

Note that the extra factor in (4.2), as compared to (4.1), is independent of k and is equal to 1 when ξ is equal to 0. This extra factor, therefore, does not affect the leading order asymptotics. On the other hand, this extra factor can become unbounded near the unstable set which means that pointwise behavior of r tends to be worse than that of s .

Let $\Lambda : \mathfrak{g} \times \Phi^{-1}(0) \rightarrow M_s$ denote the diffeomorphism $\Lambda(\xi, x_0) = e^{i\xi} \cdot x_0$ of Theorem 2.3. Recall also that we have chosen an orthonormal basis $\Xi = \{\xi_j\}_{j=1}^d$ of \mathfrak{g} to which there corresponds a Lebesgue measure $d^d\xi$ on \mathfrak{g} . The Liouville volume $\varepsilon_\omega = \omega^{\wedge n}/n!$ on M (which is the same as the Riemannian volume) decomposes as

$$\Lambda^*(\varepsilon_\omega)_{(\xi, x_0)} = \tau(\xi, x_0) d^d\xi \wedge d\text{vol}(\Phi^{-1}(0))_{x_0}$$

for some G -invariant smooth Jacobian function $\tau \in C^\infty(\mathfrak{g} \times \Phi^{-1}(0))$. We are now ready to state the remaining main results of this section. Let $\gamma_\xi : [0, 1] \rightarrow M_s$ be the path $\gamma_\xi(t) = e^{it\xi} \cdot x_0$; then $\int_{\gamma_\xi} \phi_\xi = \int_0^1 \phi_\xi(e^{it\xi} \cdot x_0) dt$.

Theorem 4.2. *Suppose s is a G -invariant holomorphic section of $\ell^{\otimes k}$. Then the norm of s can be computed by*

$$\|s\|^2 := (k/2\pi)^{n/2} \int_M |s|^2 \varepsilon_\omega = (k/2\pi)^{(n-d)/2} \int_{M//G} |A_k s|^2 I_k \varepsilon_{\widehat{\omega}} \quad (4.3)$$

where, with $[x_0] = G \cdot x_0$,

$$I_k([x_0]) = \text{vol}(G \cdot x_0) (k/2\pi)^{d/2} \int_{\mathfrak{g}} \tau(\xi, x_0) \exp \left\{ -2k \int_{\gamma_\xi} \phi_\xi \right\} d^d\xi. \quad (4.4)$$

In the case of a Hamiltonian torus action (i.e., the case when G is commutative), a similar formula was obtained by Charles in [Char06b, Sec 4.5]. Nevertheless, our result is slightly stronger than that of Charles, in that Charles inserts into the first integral in (4.3) a function f that is assumed to be supported away from the unstable set. In general, this allows him to obtain asymptotics about Toeplitz operators (as we do also in Theorem 5.4). To obtain results about the norm of a section, Charles takes f to be identically equal to one in a neighborhood of the zero-set, but still supported away from the unstable set. The insertion of such a cutoff function leaves unchanged the asymptotics to all orders of the first integral in (4.3). Still, our formula actually gives an exact (not just asymptotic) expression for the norm of an invariant section upstairs as an integral over the downstairs manifold. To get this, we require some estimates for the blow-up of certain Jacobians as we approach the unstable set.

Theorem 4.3. *Suppose r is a G -invariant holomorphic section of $\ell^{\otimes k} \otimes \sqrt{K}$. Then the norm of r can be computed by*

$$\|r\|^2 := (k/2\pi)^{n/2} \int_M |r|^2 \varepsilon_\omega = (k/2\pi)^{(n-d)/2} \int_{M//G} |B_k r|^2 J_k \varepsilon_{\tilde{\omega}}$$

where

$$J_k([x_0]) = (k/2\pi)^{d/2} 2^{d/2} \int_{\mathfrak{g}} \tau(\xi, x_0) \exp \left\{ - \int_{\gamma_\xi} \left(2k \phi_\xi + \frac{\mathcal{L}_{JX^\xi} \varepsilon_\omega}{2\varepsilon_\omega} \right) \right\} d^d \xi. \quad (4.5)$$

Note the volume factor that is present in the expression for I_k but not in the expression for J_k . In computing I_k , we decompose M_s as $\mathfrak{g} \times \Phi^{-1}(0)$ and then integrate out the ξ -dependent part of (4.1), leaving an integration over $\Phi^{-1}(0)$. Since the resulting integrand on $\Phi^{-1}(0)$ will be G -invariant, the integration over $\Phi^{-1}(0)$ can be turned into an integration over $\Phi^{-1}(0)/G$. The volume factor in (4.4) arises because the projection map $\pi : \Phi^{-1}(0) \rightarrow \Phi^{-1}(0)/G$ is a Riemannian submersion, whence the Riemannian volume measure on $\Phi^{-1}(0)$ maps to the Riemannian volume measure on $\Phi^{-1}(0)/G$, *multiplied by* a density given by the volume factor. This crucial fact is the basic reason that the Guillemin–Sternberg map (without half-forms) is *not* asymptotically unitary.

Meanwhile, the volume factor *fails* to arise in (4.5) because it is canceled by the volume factor in Theorem 3.3, which relates the pointwise magnitude of r at x_0 to the pointwise magnitude of $B_k r$ at $[x_0]$.

In Section 5, we will calculate the asymptotic behavior of the expressions for densities I_k and J_k as k tends to infinity. The calculation is done by means of Laplace’s method and uses the Hessian computed in Theorem 4.1. Because the extra factor in the expression for $|r|^2(e^{i\xi} \cdot x_0)$ is independent of k and equal to 1 at $\xi = 0$, this factor does not affect the leading-order asymptotics of the norm. The different asymptotic behavior of the maps A_k and B_k is due to the volume factor in the expression for I_k that is not present in the expression for J_k .

If it should happen that I_k is constant, then A_k (which is in any case invertible for all k) will be a constant multiple of a unitary map. Similarly, if J_k is constant for some large k (large enough that B_k is invertible), then B_k will be a constant multiple of a unitary map. There does not, however, seem to be any reason that either I_k or J_k should typically be constant. Nevertheless, we will see in Section 5 that J_k asymptotically approaches 1 for large k , which implies that B_k is asymptotically unitary. On the other hand, I_k is not asymptotically constant unless all the G -orbits in $\Phi^{-1}(0)$ happen to have the same volume. In the case where the G -orbits do not all have the same volume, it is not hard to show (Theorem 5.3) that A_k is not asymptotic to any constant multiple of a unitary map.

A similar analysis shows that if we consider a Toeplitz operator with a G -invariant symbol f upstairs, then the matrix entries for such an operator can be expressed an integral over the downstairs manifold involving a certain density $I_k(f)$, which reduces to I_k when $f \equiv 1$. Theorem 5.4 shows that in the case with half forms we obtain an asymptotic equivalence of a very simple form between Toeplitz operators upstairs and downstairs.

We now turn to the proofs of Theorems 4.1 and 4.2. First, we will decompose the integral over M_s into integrals over \mathfrak{g} and $\Phi^{-1}(0)$ and the integral over $\Phi^{-1}(0)$ into integrals over $M//G$ and $G \cdot x_0$. Both of these integral decompositions follow from the coarea formula [Chav06, pp. 159–160]:

Lemma 4.4 (Coarea formula). *Let Q and N be smooth Riemannian manifolds with $\dim Q \geq \dim N$, and let $p : Q \rightarrow N$. Then for any $f \in L^1(M)$ one has*

$$\int_Q \mathcal{J}_p f \, d\text{vol}(Q) = \int_N d\text{vol}(N)(y) \int_{p^{-1}(y)} (f|_{p^{-1}(y)}) \, d\text{vol}(p^{-1}(y))$$

where the Jacobian is $\mathcal{J}_p := \sqrt{\det p_* \circ p_*^{\text{adj}}}$.

Applying the coarea formula to the map $\text{pr}_2 \circ \Lambda^{-1} : e^{i\xi} \cdot x_0 \mapsto x_0$ from M_s to M_0 , and identifying $\exp(i\mathfrak{g}) \cdot x_0$ with \mathfrak{g} , we have that for every $f \in L^1(M_s)$

$$\int_{M_s} f \, d\text{vol}(M_s) = \int_{M_0} \int_{\mathfrak{g}} (f|_{\exp(i\mathfrak{g}) \cdot x_0}) \tau \, d^d \xi \, d\text{vol}(M_0)$$

where τ is the Jacobian \mathcal{J} of the map $\Lambda : \mathfrak{g} \times \Phi^{-1}(0) \rightarrow M_s$.

Since the volume form on the quotient $M//G$ is $d\text{vol}(M//G) = \widehat{\omega}^{n-d}/(n-d)!$ and since $\Phi^{-1}(0) \rightarrow M//G$ is a Riemannian submersion, we have [GLP99, Sec. 2.2]

$$d\text{vol}(\Phi^{-1}(0)) = d\text{vol}(G \cdot x_0) \wedge \frac{\pi_{hor}^* \widehat{\omega}^{n-d}}{(n-d)!}$$

where $\pi_{hor}^* \widehat{\omega}^{n-d}$ denotes the horizontal lift (i.e., pullback composed with projection to the horizontal subspace $W_{x_0}^{\perp B}$) of the Liouville form on the quotient. The 2-form $\pi^* \widehat{\omega} = i^* \omega$ is already horizontal, though, since

$$i(X^\xi) \pi^* \widehat{\omega} = i(X^\xi) i^* \omega = d\phi_\xi|_{\Phi^{-1}(0)} = 0.$$

This proves the following.

Lemma 4.5. *Let $\pi : \Phi^{-1}(0) \rightarrow M//G := \Phi^{-1}(0)/G$ denote the canonical projection. Then $\mathcal{J}_\pi = 1$; in particular, for every G -invariant function $f \in L^1(\Phi^{-1}(0))^G$ we have*

$$\int_{\Phi^{-1}(0)} f \, d\text{vol}(\Phi^{-1}(0)) = \int_{M//G} \text{vol}(G \cdot x_0) f([x_0]) \, d\text{vol}(M//G).$$

To prove Theorem 4.1, we first make the following computations:

Lemma 4.6. *The first two derivatives of the norm of a G -invariant holomorphic section $s \in \mathcal{H}(M; \ell^{\otimes k})^G$ along γ_ξ are:*

$$\begin{aligned} JX^\xi |s|^2 &= -2k \phi_\xi |s|^2, \text{ and} \\ JX^{\xi_1} JX^{\xi_2} |s|^2 &= -2k B(X^{\xi_1}, X^{\xi_2}) |s|^2 + 4k^2 \phi_{\xi_1} \phi_{\xi_2} |s|^2. \end{aligned}$$

The last statement of Theorem 4.1 follows immediately upon combining the two above equations and restricting to the zero-set.

Proof. Since the connection on $\ell^{\otimes k}$ is Hermitian, we have

$$JX^\xi |s|^2 = (\nabla_{JX^\xi} s, s) + (s, \nabla_{JX^\xi} s).$$

By assumption, s is G -invariant. Since the action of G on $\mathcal{H}(M; \ell^{\otimes k})$ is given by equation (2.6), we have

$$0 = Q_\xi s = (\nabla_{X^\xi} - ik \phi_\xi) s$$

so that $\nabla_{X^\xi} s = -ik \phi_\xi s$. The projection of a vector field onto its $(0, 1)$ -part is $X \mapsto \frac{1}{2}(1 - iJ)X$. We assume that s is holomorphic, i.e., $\nabla_{(1-iJ)X^\xi} s = 0$. Therefore $\nabla_{JX^\xi} s = i\nabla_{X^\xi} s$. Putting this all together, we get

$$JX^\xi(s, s) = -i(ik \phi_\xi s, s) + i(s, ik \phi_\xi s) = -2k \phi_\xi(s, s).$$

Next, the Leibniz rule yields the second derivative:

$$JX^{\xi_j} JX^{\xi_l} |s|^2 = -2k JX^{\xi_j}(\phi_{\xi_l}) + 4k^2 \phi_{\xi_l} \phi_{\xi_j} |s|^2.$$

Since $d\phi_{\xi_i} = i(X^{\xi_i})\omega$ we have

$$JX^{\xi_j}\phi_{\xi_i} = \omega(X^{\xi_i}, JX^{\xi_j}) = B(X^{\xi_j}, X^{\xi_i}).$$

Therefore,

$$JX^{\xi_j}JX^{\xi_i}|s|^2 = -2k B(X^{\xi_j}, X^{\xi_i})|s|^2 + 4k^2 \phi_{\xi_j}\phi_{\xi_i}|s|^2$$

as desired. \square

Proof of Theorem 4.1(a). Applying Lemma 4.6, the derivative of $|s|^2$ along γ_ξ is

$$\frac{d}{dt}\gamma_\xi^*|s|^2(t) = JX^\xi|s|^2(e^{it\xi} \cdot x_0) = -2k\phi_\xi(e^{it\xi} \cdot x_0)|s|^2(e^{it\xi} \cdot x_0).$$

so that

$$\frac{d}{dt}\log \gamma_\xi^*|s|^2(t) = -2k\phi(e^{it\xi} \cdot x_0).$$

Integrating this equation from $t = 0$ to $t = 1$ yields the desired result. \square

As Guillemin and Sternberg pointed out, an important consequence of Theorem 4.1 is that a G -invariant holomorphic $s \in \mathcal{H}(M; \ell^{\otimes k})$ attains its (unique) maximum value in the orbit $G_{\mathbb{C}} \cdot x_0$ on the zero-set of the moment map $(G_{\mathbb{C}} \cdot x_0) \cap \Phi^{-1}(0) = G \cdot x_0$. The reason is that JX^ξ is the gradient vector of the function ϕ_ξ , since

$$B(JX^\xi, \cdot) = \omega(JX^\xi, J\cdot) = \omega(X^\xi, \cdot) = d\phi_\xi. \quad (4.6)$$

Hence, ϕ_ξ is increasing along γ_ξ and so $|s|^2$ is decreasing away from $\Phi^{-1}(0)$.

A useful and direct consequence of this fact is that each $G_{\mathbb{C}}$ -orbit intersects the zero-set in exactly one G -orbit; since if not, then there exist points $x, x' \in \Phi^{-1}(0)$ which lie in distinct G -orbits such that $x' = e^{i\xi} \cdot x$ for some $\xi \in \mathfrak{g}$. But this is impossible since moment map is strictly increasing along the path $t \mapsto e^{it\xi} \cdot x$.

Next we express the norm of a G -invariant half-form corrected section $r \in \mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{K})^G$ along γ_ξ in terms of its value at x_0 .

Proof of Theorem 4.1(b). Locally, a section $r \in \mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{K})^G$ can be written as $r = s\nu$ for some G -invariant holomorphic half-form ν . Since ν is a G -invariant holomorphic half-form, the combination $\nu^2 \wedge \bar{\nu}^2$ is $G_{\mathbb{C}}$ -invariant, whence

$$0 = \mathcal{L}_{JX^\xi}(\nu^2 \wedge \bar{\nu}^2) = \mathcal{L}_{JX^\xi}((\nu, \nu)^2 \varepsilon_\omega) = [2(\nu, \nu) JX^\xi(\nu, \nu)] \varepsilon_\omega + (\nu, \nu)^2 \mathcal{L}_{JX^\xi} \varepsilon_\omega$$

so that

$$\frac{d}{dt}\log \gamma_\xi^*(\nu, \nu)(t) = \frac{JX^\xi(\nu, \nu)}{(\nu, \nu)}(e^{it\xi} \cdot x_0) = -\frac{\mathcal{L}_{JX^\xi} \varepsilon_\omega}{2\varepsilon_\omega}(e^{it\xi} \cdot x_0).$$

Integrating this along γ_ξ and combining it with Theorem 4.1(a) yields the desired result. \square

By decomposing the Liouville measure on M_s in terms of the global decomposition $\psi : \mathfrak{g} \times \Phi^{-1}(0) \rightarrow M_s$ (Theorem 2.3) and the fibration $G \rightarrow \Phi^{-1}(0) \rightarrow M//G$, we will obtain our desired integral. Recall that we have chosen a basis $\Xi = \{\xi_j\}_{j=1}^d$ of \mathfrak{g} to which there corresponds a Lebesgue measure $d^d\xi$ on \mathfrak{g} .

The Liouville volume $\varepsilon_\omega = \omega^{\wedge n}/n!$, which is the same as the Riemannian volume, decomposes as

$$\Lambda^*(\varepsilon_\omega)_{(\xi, x_0)} = \tau(\xi, x_0) d^d\xi \wedge d\text{vol}(\Phi^{-1}(0))_{x_0} \quad (4.7)$$

for some G -invariant smooth Jacobian function $\tau \in C^\infty(\mathfrak{g} \times \Phi^{-1}(0))$ (this is just the coarea formula (4.4) applied to the map $M_s \rightarrow M_0$ given by $e^{i\xi} \cdot x_0 \mapsto x_0$).

Proof of Theorem 4.2. Combining (4.7) and Theorem 4.1 yields

$$\begin{aligned}
(k/2\pi)^{n/2} \int_M |s|^2 \varepsilon_\omega &= (k/2\pi)^{(n-d)/2} \int_{M_0} (k/2\pi)^{d/2} \int_{\mathfrak{g}} |\pi^* A_k s|^2(x_0) \exp \left\{ -2k \int_{\gamma_\xi} \phi_\xi \right\} \\
&\quad \times \tau(\xi, x_0) d^d \xi d\text{vol}(M_0) \quad (4.8)
\end{aligned}$$

since $s(x_0) = \pi^* A_k s(x_0)$. The terms $|\pi^* A_k s|^2$ and τ are obviously G -invariant. To see that the exponential factor in the integrand is G -invariant, define

$$\rho(e^{i\xi} \cdot x_0) = 2 \int_{\gamma_\xi} \phi_\xi = 2 \int_0^1 \langle \Phi(e^{it\xi} \cdot x_0), \xi \rangle dt.$$

To evaluate $g^* \rho$, we rewrite $ge^{i\xi} \cdot x_0 = e^{i\text{Ad}(g)\xi} g \cdot x_0$. Then

$$\begin{aligned}
g^* \rho(e^{i\xi} \cdot x_0) &= \rho(e^{i\text{Ad}(g)\xi} g \cdot x_0) = 2 \int_0^1 \langle \Phi(e^{i\text{Ad}(g)\xi t} g \cdot x_0), \text{Ad}(g)\xi \rangle dt \\
&= 2 \int_0^1 \langle \Phi(e^{it\xi} \cdot x_0), \xi \rangle dt = \rho(e^{i\xi} \cdot x_0)
\end{aligned}$$

where we have used the G -equivariance of the moment map $\Phi(gp) = \text{Ad}^*(g^{-1})\Phi(p)$. Hence, the integrand in (4.8) is G -invariant, so by the coarea formula (Lemma 4.5) the integral over M_0 reduces to an integral over $M//G$ with an extra factor of $\text{vol}(G \cdot x_0)$, and we obtain the desired result. \square

We now compute the norm of a G -invariant holomorphic section as an integral over the zero-set of the moment map taking into account the metaplectic correction.

Proof of Theorem 4.3. The proof is the similar to the above proof of Theorem 4.2; we first write the integral over M as a multiple integral:

$$\begin{aligned}
(k/2\pi)^{n/2} \int_M |r|^2 \varepsilon_\omega &= (k/2\pi)^{(n-d)/2} \int_{M_0} (k/2\pi)^{d/2} \\
&\quad \times \int_{\mathfrak{g}} |r|^2(x_0) \exp \left\{ - \int_{\gamma_\xi} (2k \phi_\xi + \mathcal{L}_{JX\xi} \varepsilon_\omega / 2\varepsilon_\omega) \right\} \\
&\quad \times \tau(\xi, x_0) d^d \xi d\text{vol}(M_0).
\end{aligned}$$

The main difference from the proof of Theorem 4.2 is that by Theorem 3.3 we have

$$|r|^2(x_0) = 2^{d/2} \text{vol}(G \cdot x_0)^{-1} |B_k r|^2([x_0]).$$

Inserting this into the above integral, and noting that the remaining integrand is G -invariant (following the same argument as in the proof of Theorem 4.2) we obtain (again using the coarea formula Lemma 4.4) that

$$\begin{aligned}
(k/2\pi)^{n/2} \int_M |r|^2 \varepsilon_\omega &= (k/2\pi)^{(n-d)/2} \int_{M//G} |r|^2([x_0]) \\
&\quad \times (k/2\pi)^{d/2} 2^{d/2} \int_{\mathfrak{g}} \exp \left\{ - \int_{\gamma_\xi} (2k \phi_\xi + \frac{\mathcal{L}_{JX\xi} \varepsilon_\omega}{2\varepsilon_\omega}) \right\} \\
&\quad \times \tau(\xi, x_0) d\text{vol}(\exp(i\mathfrak{g}) \cdot x_0) d^d \xi \varepsilon_\omega
\end{aligned}$$

whence

$$\|r\|^2 = (k/2\pi)^{(n-d)/2} \int_{M//G} |B_k r|^2([x_0]) J_k([x_0]) \varepsilon_{\tilde{\omega}}$$

as desired. \square

5 Asymptotics

In this section we compute the leading order asymptotics of the densities

$$I_k(f)([x_0]) = \text{vol}(G \cdot x_0) (k/2\pi)^{d/2} \int_{\mathfrak{g}} \tau(\xi, x_0) f(x_0, \xi) \exp \left\{ -2k \int_{\gamma_\xi} \phi_\xi \right\} d^d \xi,$$

and

$$J_k(f)([x_0]) = (k/2\pi)^{d/2} 2^{d/2} \int_{\mathfrak{g}} \tau(\xi, x_0) f(x_0, \xi) \exp \left\{ - \int_{\gamma_\xi} \left(2k \phi_\xi + \frac{\mathcal{L}_{JX^\xi \varepsilon_\omega}}{2\varepsilon_\omega} \right) \right\} d^d \xi,$$

where f is a smooth, G -invariant function on M . In the case that $f \equiv 1$, these asymptotics will give asymptotics of the Guillemin–Sternberg-type maps. For arbitrary smooth $f \in C^\infty(M)^G$, the asymptotics will give information about Toeplitz operators.

The main result is the following.

Theorem 5.1. *For $f \in C^\infty(M)^G$, the densities $I_k(f)$ and $J_k(f)$ satisfy*

$$\begin{aligned} \lim_{k \rightarrow \infty} I_k(f)([x_0]) &= 2^{-d/2} f(x_0) \text{vol}(G \cdot x_0), \text{ and} \\ \lim_{k \rightarrow \infty} J_k(f)([x_0]) &= f(x_0) \end{aligned}$$

for each $x_0 \in \Phi^{-1}(0)$, and the limits are uniform.

Our main tool will be Laplace’s approximation [BIH75, Chap. 10], also frequently referred to as the stationary phase approximation or the method of steepest descent. Let $D \subset \mathbb{R}^d$ be a bounded domain, and consider $\rho \in C^2(D)$ and $\sigma \in C(D)$. Suppose ρ attains a unique minimum at $x_0 \in D \setminus \partial D$ (i.e., the interior of D). Laplace’s approximation gives the leading order asymptotic limit

$$I(k) = \int_D \sigma(x) e^{-k\rho(x)} d^d x \sim e^{k\rho(x_0)} \left(\frac{2\pi}{k} \right)^{d/2} |\det H_\rho(x_0)|^{-1/2} \sigma(x_0), \quad k \rightarrow \infty, \quad (5.1)$$

where H_ρ denotes the Hessian of ρ .

The formula for the large- k limits of I_k and J_k come from applying Laplace’s method to the integral over $\mathfrak{g} \simeq \mathbb{R}^d$ in Theorems 4.2 and 4.3, using the computation of the relevant Hessian, which we have already performed in Theorem 4.1. Conceptually, then, it is easy to understand where the limiting formulas come from.

There are, however, some technicalities to attend to, and it is these technicalities that will occupy the bulk of this section. First, we need to consider the part of the integrals near infinity as well as the part near the origin. The density $\tau(x_0, \xi)$ can (apparently) blow up as ξ tends to infinity for certain values of x_0 . As a result, we need some estimates (Lemma 5.7) to ensure that the ξ -integrals in (4.4) or (4.5) are finite for all (as opposed to almost all) x_0 , at least for large k . Second, we wish to be careful in verifying that the limits are uniform over $M//G$, which is needed to obtain the asymptotic unitarity of the maps B_k .

Before coming to these technicalities we state the consequences of the above asymptotic formulas for the unitarity of the maps A_k and B_k and indicate some applications to the asymptotics of Toeplitz operators.

Theorem 5.2. *The maps B_k are asymptotically unitary, in the sense that*

$$\lim_{k \rightarrow \infty} \|B_k^* B_k - I\| = \lim_{k \rightarrow \infty} \|B_k B_k^* - I\| = 0,$$

where $\|\cdot\|$ refers to the operator norm.

Proof. Let $\hat{r} \in \mathcal{H}(M//G; \hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}})$. Then for all $t \in \mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{K})$ we have

$$\langle B_k^* \hat{r}, t \rangle_M = \langle \hat{r}, B_k t \rangle_{M//G}.$$

On the other hand, Theorem 4.3 implies

$$\langle B_k^* \hat{r}, t \rangle_M = \langle J_k B_k B_k^* \hat{r}, B_k t \rangle_{M//G}, \quad \forall t \in \mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{K}).$$

Since B_k is bijective (for k sufficiently large), every $\hat{t} \in \mathcal{H}(M//G; \hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}})$ is of the form $B_k t$ for some $t \in \mathcal{H}(M; \ell^{\otimes k} \otimes \sqrt{K})$; so for all $\hat{t} \in \mathcal{H}(M//G; \hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}})$ we have

$$\langle \hat{r}, \hat{t} \rangle_{M//G} = \langle J_k B_k B_k^* \hat{r}, \hat{t} \rangle_{M//G}.$$

Applying Theorem 5.1 then yields the claim that B_k is asymptotically unitary:

$$\lim_{k \rightarrow \infty} \|B_k^* B_k - I\| = \lim_{k \rightarrow \infty} \|\text{mult. by } (J_k^{-1} - 1)\| \leq \lim_{k \rightarrow \infty} \max_{M//G} |J_k^{-1} - 1| = 0.$$

□

Theorem 5.3. *If $\text{vol}(G \cdot x_0)$ is not constant on $M//G$, then there is no sequence c_k of constants for which $\|A_k A_k^* - c_k I\|$ tends to zero as k tends to infinity.*

In the torus case, a similar result was proved by Charles in [Char06b, Remark 4.29].

Proof. This theorem follows easily from the existence of localized sections (see [MM04, pp. 36–37]). For each $[x] \in M//G$, there exists a sequence of holomorphic sections $\{S_{[x]}^{(k)} \in \mathcal{H}(M//G; \hat{\ell}^{\otimes k})\}$ which are asymptotically concentrated near $[x]$ in the sense that for every $f \in C(M//G)$

$$\lim_{k \rightarrow \infty} (k/2\pi)^{(n-d)/2} \int_{M//G} f |S_{[x]}^{(k)}|^2 \varepsilon_{\widehat{\omega}} = f([x]).$$

If $\text{vol}(G \cdot x_0)$ is not constant on $M//G$, there exist points $[x_{\max}]$ and $[x_{\min}]$ where $\text{vol}(G \cdot x_0)$ achieves its maximum and minimum values. Consider a sequence of peak sections localized at $[x_{\max}]$. Let $\{c_k\}$ be a sequence of constants. Then

$$\|A_k A_k^* - c_k I\|^2 \geq \|A_k A_k^* S_{[x_{\max}]}^{(k)} - c_k S_{[x_{\max}]}^{(k)}\|^2 \rightarrow |I_k^{-1}([x_{\max}]) - c_k|^2, \quad k \rightarrow \infty.$$

Similarly, we obtain

$$\|A_k A_k^* - c_k I\|^2 \geq \|A_k A_k^* S_{[x_{\min}]}^{(k)} - c_k S_{[x_{\min}]}^{(k)}\|^2 \rightarrow |I_k^{-1}([x_{\min}]) - c_k|^2, \quad k \rightarrow \infty.$$

Since

$$I_k([x_{\max}]) \rightarrow \text{vol}(G \cdot x_{\max}) \neq \text{vol}(G \cdot x_{\min}) \leftarrow I_k([x_{\min}]), \quad k \rightarrow \infty,$$

we cannot have both

$$|I_k^{-1}([x_{\max}]) - c_k|^2 \rightarrow 0 \quad \text{and} \quad |I_k^{-1}([x_{\min}]) - c_k|^2 \rightarrow 0.$$

Hence $\|A_k A_k^* - c_k I\|^2 \not\rightarrow 0$.

□

The asymptotic estimates of Theorem 5.1 can be used to derive results about Toeplitz operators with G -invariant symbols. We mention here the nicest result, concerning the asymptotics of Toeplitz operators in the case where half-forms are included. Let $f \in C^\infty(M)$, and recall that the Toeplitz operator with symbol f is the map T_f from $\mathcal{H}(M, \ell^{\otimes k} \otimes \sqrt{K})$ to itself defined by

$$T_f s = \text{proj}(f s),$$

where “proj” denotes the orthogonal projection from the space of all square-integrable sections onto the holomorphic subspace. If f is G -invariant, then precisely the same argument as in the proof of Theorem 4.3 shows that for $r_1, r_2 \in \mathcal{H}(M, \ell^{\otimes k} \otimes \sqrt{K})$ we have

$$\langle r_1, T_f r_2 \rangle = (k/2\pi)^{n/2} \int_M f \cdot (r_1, r_2) \varepsilon_\omega = (k/2\pi)^{(n-d)/2} \int_{M//G} (B_k r_1, B_k r_2) J_k(f) \varepsilon_{\hat{\omega}}.$$

If we denote by \hat{T}_ϕ the Toeplitz operator on $\mathcal{H}(M//G, \hat{\ell}^{\otimes k} \otimes \sqrt{\hat{K}})$ with symbol $\phi \in C^\infty(M//G)$, then the above formula becomes

$$\langle r_1, T_f r_2 \rangle = \left\langle B_k r_1, \hat{T}_{J_k(f)} B_k r_2 \right\rangle.$$

The asymptotics of $J_k(f)$ then immediately imply the following.

Theorem 5.4. *Let $f \in C^\infty(M)^G$ and let \hat{f} denote the restriction of f to $\Phi^{-1}(0)$, regarded as a function on $\Phi^{-1}(0)/G$. Let T_f^G denote the restriction of the Toeplitz operator T_f to the space of G -invariant sections. Then T_f^G is asymptotically equivalent to $\hat{T}_{\hat{f}}$ in the sense that*

$$\left\| \hat{T}_{\hat{f}} - B_k \circ T_f^G \circ B_k^{-1} \right\| \rightarrow 0$$

as $k \rightarrow \infty$.

Since we also prove that the operators B_k are asymptotically unitary, we may say that T_f^G and $\hat{T}_{\hat{f}}$ are “asymptotically unitarily equivalent.” Such a result is certainly does *not* hold without half-forms, indicating again the utility of the metaplectic correction.

We now turn to the proof of Theorem 5.1.

Proof. For all $\hat{r}, \hat{t} \in \mathcal{H}(M//G, \hat{\ell}^{\otimes k} \otimes \sqrt{\hat{K}})$, we have

$$\left\langle \hat{r}, (\hat{T}_{\hat{f}} - B_k \circ T_f^G \circ B_k^{-1}) \hat{t} \right\rangle = \left\langle \hat{r}, (\hat{f} - J_k(f)) \hat{t} \right\rangle.$$

Hence, $\left\| \hat{T}_{\hat{f}} - B_k \circ T_f^G \circ B_k^{-1} \right\| = \left\| \text{mult. by } (\hat{f} - J_k(f)) \right\|$. It then follows from Theorem 5.1 that

$$\lim_{k \rightarrow \infty} \left\| \hat{T}_{\hat{f}} - B_k \circ T_f^G \circ B_k^{-1} \right\| = \lim_{k \rightarrow \infty} \left\| \text{mult. by } (\hat{f} - J_k(f)) \right\| \leq \lim_{k \rightarrow \infty} \max_{M//G} |\hat{f} - J_k(f)| = 0.$$

□

We now turn to the proof of Theorem 5.1. Our first task will be to control the part of the integral near infinity. From the definition of τ , it is apparent that the integral of $\tau(\xi, x_0)$ over \mathfrak{g} is finite for *almost all* x_0 , but it is not obvious that this integral is finite for *all* x_0 . We will, however, give uniform exponential bounds on the behavior of $\tau(\xi, x_0)$ as ξ tends to infinity. This is sufficient to show that $\tau(\xi, x_0) \exp \left\{ -2k \int_{\gamma_\xi} \phi_\xi \right\}$ is finite for all x_0 , provided k is sufficiently large, and that the part of the integral outside a neighborhood of the origin is uniformly negligible as k tends to infinity.

Our second task will be to show that Laplace's approximation can be applied so as to give *uniform* limits, which we need to prove asymptotic unitarity of B_k . Theorems 4.2 and 4.3 express the densities I_k and J_k in the form of (5.1), where in the case of the expression for J_k , the k -independent term in the exponent should be factored out and grouped with $\tau(\xi, x_0)$. We develop a uniform version of Laplace's approximation that allows for the desired uniformity. (It is probably well known that this is possible, but we were not able to find a written proof.) Our argument will be based on the Morse–Bott Lemma (a parameterized version of the usual Morse lemma).

Once these two tasks are accomplished, it is a straightforward matter to plug in the Hessian computation in Theorem 4.1 to obtain Theorem 5.1.

5.1 Growth estimates

In this section, we show that the contribution to I_k coming from the complement of a tubular neighborhood of the zero-set is negligible as k tends to infinity:

Theorem 5.5. *There exist constants $b, D > 0$ such that for all $x_0 \in \Phi^{-1}(0)$ and for all R and k sufficiently large*

$$\int_{\mathfrak{g} \setminus B_R(0)} f(\xi, x_0) \tau(\xi, x_0) \exp \left\{ - \int_{\gamma_\xi} 2k \phi_\xi \right\} d^d \xi \leq b e^{-RDk},$$

where $B_R(0)$ is a ball of radius R centered at $0 \in \mathfrak{g}$.

Our first step is to show that the integrand appearing in I_k decays exponentially in the radial directions.

Lemma 5.6. *There exists $C > 0$ such that for all sufficiently large t ,*

$$\exp \left\{ - \int_{\gamma_{t\hat{\xi}}} 2k \phi_{t\hat{\xi}} \right\} \leq e^{-2ktC}$$

uniformly on $\Phi^{-1}(0)$, where $\hat{\xi} \in \mathfrak{g}$ with $|\hat{\xi}| = 1$.

Proof. By definition, we have

$$\int_{\gamma_{t\hat{\xi}}} \phi_{t\hat{\xi}} = \int_0^1 \langle \Phi(e^{i\tau t\hat{\xi}} \cdot x_0), t\hat{\xi} \rangle d\tau = t \int_0^1 \langle \Phi(e^{i\tau t\hat{\xi}} \cdot x_0), \hat{\xi} \rangle d\tau.$$

Hence, it is enough to show that there exists some $C > 0$ such that for t sufficiently large and for all $x_0 \in \Phi^{-1}(0)$

$$\int_0^1 \langle \Phi(e^{i\tau t\hat{\xi}} \cdot x_0), \hat{\xi} \rangle d\tau \geq C.$$

For $t \geq 0$, define

$$M_t = \left\{ e^{it\xi} \cdot x_0 : |\hat{\xi}| = 1, x_0 \in \Phi^{-1}(0) \right\}. \quad (5.2)$$

Then M_t , $t > 0$, is a sphere bundle over $\Phi^{-1}(0)$ and is hence compact. Define $\rho_t : M_t \rightarrow \mathbb{R}$ by

$$\rho_t(e^{it\hat{\xi}} \cdot x_0) = 2 \int_0^1 \phi_{\hat{\xi}}(e^{i\tau t\hat{\xi}} \cdot x_0) d\tau.$$

Fix $t > 0$. Then since ρ_t is smooth with compact domain there is some $m \in M_t$ where ρ_t achieves its minimum. Recall that $e^{it\tau\hat{\xi}} \cdot x_0$ is the gradient line of $\phi_{t\hat{\xi}}$. Moreover,

$$(\text{grad } \phi_{\hat{\xi}})_{e^{i\xi} \cdot x_0} = JX_{e^{i\xi} \cdot x_0}^{\hat{\xi}} \text{ and } |JX_{e^{i\xi} \cdot x_0}^{\hat{\xi}}| = |X_{e^{i\xi} \cdot x_0}^{\hat{\xi}}| \neq 0$$

since G acts freely on M_s . Therefore, $\phi_{t\hat{\xi}}$ is strictly increasing along $e^{it\tau\hat{\xi}} \cdot x_0$. In particular, $2C := \rho_t(m) > 0$ so that, by the definition of m , we have $\rho_t(m') \geq 2C$ for all $m' \in M_t$. Hence we obtain $\int_0^1 \langle \Phi(e^{i\tau t'} \cdot x_0), t\hat{\xi} \rangle d\tau \geq C$ for all $t' > t$ and for all $x_0 \in \Phi^{-1}(0)$. \square

We now turn to the task of estimating τ , which we will do by embedding M into projective space. We recall some basic projective geometry. The function τ is the Jacobian of the map $\psi : \mathfrak{g} \times \Phi^{-1}(0) \rightarrow M$. Since M is a compact Kähler manifold and ℓ is an equivariant ample line bundle, we can equivariantly embed M into $\mathbb{C}P^N$ for some N (this is Kodaira's embedding theorem enhanced by the presence of a G -invariant Kähler structure and line bundle; the embedding is realized by considering holomorphic sections of high tensor powers of ℓ). We identify M with its embedded image. Under this embedding, we identify G with a compact subgroup of $PU(N+1)$.

The metric on the embedded image of M induced by the Kähler metric on M differs from the Fubini–Study metric on $\mathbb{C}P^N$ by smooth factors. In particular, the difference between computing a determinant with the Fubini–Study metric and the induced Kähler metric is the determinant of the linear map which takes a Fubini–Study orthogonal basis of the tangent space to a basis which is orthogonal with respect to the induced metric. This map is invertible and smooth on M and hence is bounded and bounded away from zero, and so it suffices for our purposes to compute the Jacobian τ using the Fubini–Study metric.

We use homogeneous coordinates on $\mathbb{C}P^N$; i.e., $\mathbb{C}P^N = \{[\mathbf{z} = (z^0, z^1, \dots, z^N)] : z^j \in \mathbb{C}\}$. The tangent space $T_{[\mathbf{z}]} \mathbb{C}P^N$ can be identified with the orthogonal complement of the line in \mathbb{C}^{N+1} determined by \mathbf{z} .

The Fubini–Study metric on $\mathbb{C}P^N$ is [Zhe00, Sec. 7.4]

$$g_{\text{FS}} = \text{Re} \left(\frac{|\mathbf{z}|^2 d\bar{\mathbf{z}} \otimes d\mathbf{z} - \sum_{j,k=0}^N z^j d\bar{z}^j \otimes \bar{z}^k dz^k}{|\mathbf{z}|^4} \right).$$

Now we show that $\tau(\xi, x_0)$ grows at most exponentially in the radial directions away from the zero-set.

Lemma 5.7. *There exist constants $a, b > 0$ such that for all t sufficiently large*

$$\tau(t\hat{\xi}, x_0) \leq bt^{-d}e^{at}$$

uniformly on $\Phi^{-1}(0)$ and for all $\hat{\xi} \in \mathfrak{g}$ with $|\hat{\xi}| = 1$.

Proof. Recall that $\Xi = \{\xi_j\}_{j=1}^d$ is an orthonormal basis of $\mathfrak{g} < \mathfrak{pu}(N+1)$ with respect to an Ad-invariant inner product on \mathfrak{g} . Let $\{X_j\}_{j=1}^{2n-d}$ and $\{Y_j\}_{j=1}^{2n}$ be orthonormal bases of $T_{x_0}\Phi^{-1}(0)$ and $T_{e^{it\hat{\xi}, x_0}}M$, respectively. Write a vector in terms of its components with respect to the appropriate basis using the usual convention; for example, $v = v^j Y_j \in T_{e^{i\xi, x_0}}M$. We want to estimate

$$\tau(t\hat{\xi}, x_0) = (\det \psi_*)(t\hat{\xi}, x_0)$$

We can compute the components of the images of the basis $\Xi \cup \{X_j\}$ by

$$\psi_*(\xi_j)^l = g_{\text{FS}}(Y_l, \psi_*(\xi_j)), \text{ and } \psi_*(X_j)^l = g_{\text{FS}}(Y_l, \psi_*(X_j)). \quad (5.3)$$

There is a unique $R \in \mathfrak{su}(N+1)$ such that $e^{i\hat{\xi}}[\mathbf{z}] = [e^{iR}\mathbf{z}]$. Since R is skew-Hermitian, it has pure imaginary eigenvalues and these eigenvalues vary continuously with respect to the choice of $\hat{\xi}$. Let the eigenvalues be $-\sqrt{-1}\lambda_0, \dots, -\sqrt{-1}\lambda_N$ with $\lambda_0 \leq \dots \leq \lambda_N$. Then for generic $\mathbf{z} \in \mathbb{C}^{N+1}$, we have $|(e^{itR}\mathbf{z})^j| = O(e^{\lambda_N t})$. In order to compute the components in equation (5.3), we need to

estimate the growth of both Y_l and the pushforwards $\psi_*(\xi_j)$ and $\psi_*(X_j)$. We begin with Y_l . Let $[\mathbf{w}_t] = e^{it\hat{\xi}} \cdot x_0$. Since $Y_j = (Y_j^0, \dots, Y_j^N)$ is a unit vector, we have

$$\left(|Y_j|_{\text{FS}}^2\right)_{e^{it\hat{\xi}}x_0} = 1 = \frac{1}{|\mathbf{w}_t|^4} \operatorname{Re} \left(|\mathbf{w}_t|^2 Y_j \cdot \bar{Y}_j - \sum_{l,m=0}^N w_t^l \bar{w}_t^m \bar{Y}_j^l Y_j^m \right).$$

In particular, since $|\mathbf{w}_t|^4 = O(e^{4\lambda_N t})$ and $|w_t^l| = O(e^{\lambda_N t})$, we must have $|Y_j^l| = O(e^{\lambda_N t})$. To estimate the growth rate of the pushforward $\psi_*(X_j)$, we observe that if $X_j = [\mathbf{x}_j]$, then

$$\psi_*(X_j) = [e^{itR}\mathbf{x}_j]$$

from which we obtain

$$\psi_*(X_j)^l = \frac{1}{|\mathbf{w}_t|^4} \operatorname{Re} \left(|\mathbf{w}_t|^2 \sum_m \bar{Y}_l^m (e^{itR}\mathbf{x}_j)^m - \sum_{l,m} w_t^l \bar{w}_t^m \bar{Y}_l^l (e^{itR}\mathbf{x}_j)^m \right). \quad (5.4)$$

By the same argument as above, we see that $|(e^{itR}\mathbf{x}_j)^m| = O(e^{\lambda_N t})$. We must be a little careful with our growth estimates; if $\lambda_0 > 0$, then at worst

$$|\mathbf{w}_t|^{-1} = O(e^{-\lambda_0 t}) = O(1),$$

but if $\lambda_0 < 0$, then there may be certain points x_0 where $|\mathbf{w}_t|^{-1} = O(e^{-\lambda_0 t})$, which is exponential *growth*. With this in mind, we can use equation (5.4) to make our first growth estimates:

$$\left| \psi_*(X_j)_{e^{it\hat{\xi}}x_0}^l \right| = O(e^{4(\lambda_N - \lambda_0)t}).$$

Of course, it is clear from equation (5.4) that at generic points in generic directions, the growth will actually be much less than this estimate indicates since the numerator and denominator will both grow at the same rate. Finally we consider the components $\psi_*(\xi_j)^l$. Since we have embedded M in $\mathbb{C}P^N$, the exponential e^{itR} is a matrix exponential, and there is an explicit formula for its derivative. Write $x_0 = [\mathbf{z}]$ and let $S_j \in \mathfrak{su}(N+1)$ be the matrix such that $e^{i(t\hat{\xi} + s\xi_j)} \cdot x_0 = [e^{i(tR + sS_j)}\mathbf{z}]$. Using the formula for the derivative of the exponential, we compute

$$\psi_*(\xi_j) = \frac{d}{ds} \Big|_{s=0} e^{i(t\hat{\xi} + s\xi_j)} \cdot x_0 = \left[e^{itR} \left(\frac{1 - e^{-it \operatorname{ad}(R)}}{t \operatorname{ad}(R)} \right) S_j \mathbf{z} \right].$$

As a linear operator on $\mathfrak{su}(N+1)$, the matrix $\operatorname{ad}(R)$ is skew and hence has purely imaginary eigenvalues (which vary continuously with respect to choice of ξ). Denote them by $\sqrt{-1}\mu_0, \dots, \sqrt{-1}\mu_d$ with $\mu_0 \leq \dots \leq \mu_d$. Then

$$\left| \left(e^{itR} \left(\frac{1 - e^{-it \operatorname{ad}(R)}}{t \operatorname{ad}(R)} \right) S_j \mathbf{z} \right)^m \right| = O \left(\frac{e^{(\lambda_N + \mu_d)t}}{t} \right).$$

Finally, we can use this to estimate the growth of the component $\psi_*(\xi_j)^l$ by computing with the Fubini–Study metric as above:

$$\begin{aligned} \psi_*(\xi_j)^l &= \frac{1}{|\mathbf{w}_t|^4} \left(|\mathbf{w}_t|^2 \sum_m \bar{Y}_l^m \left(e^{itR} \left(\frac{1 - e^{-it \operatorname{ad}(R)}}{t \operatorname{ad}(R)} \right) S_j \mathbf{z} \right)^m \right. \\ &\quad \left. - \sum_{l,m} w_t^l \bar{Y}_l^m \bar{w}_t^m \left(e^{itR} \left(\frac{1 - e^{-it \operatorname{ad}(R)}}{t \operatorname{ad}(R)} \right) S_j \mathbf{z} \right)^m \right) \\ &= O \left(\frac{e^{(4\lambda_0 + 3\lambda_N + \mu_d)t}}{t} \right). \end{aligned}$$

Putting this all together, we obtain an estimate of the growth of the Jacobian τ . Define

$$a(\hat{\xi}) = d(4\lambda_0 + 3\lambda_N + \mu_d) + 4(2n - d)(\lambda_N - \lambda_0).$$

Then for all $x_0 \in \Phi^{-1}(0)$

$$\begin{aligned} |\tau(t\hat{\xi}, x_0)| &= O\left(\left(t^{-1}e^{(4\lambda_0+3\lambda_N+\mu_d)t}\right)^d \left(e^{4(\lambda_N-\lambda_0)t}\right)^{2n-d}\right) \\ &= O\left(t^{-d}e^{(d(4\lambda_0+3\lambda_N+\mu_d)+4(2n-d)(\lambda_N-\lambda_0))t}\right) = O\left(t^{-d}e^{a(\hat{\xi})t}\right). \end{aligned}$$

Since the eigenvalues λ_j and μ_j depend continuously on the choice of ξ , and the unit sphere $\{\hat{\xi} \in \mathfrak{g} : \hat{\xi} = 1\}$ is compact, there is some point $\hat{\xi}_{\max} \in \mathfrak{g}$ where $a(\hat{\xi})$ attains its maximum; let $a = a(\hat{\xi}_{\max})$. Then $|\tau(t\hat{\xi}, x_0)| = O(t^{-d}e^{at})$ uniformly in x_0 and $\hat{\xi}$ as desired. \square

Next, we combine the previous two lemmas and integrate over the complement of a neighborhood of $0 \in \mathfrak{g}$ to prove Theorem 5.5.

Proof of Theorem 5.5. Introduce polar coordinate $\xi = (t, \hat{\xi})$, $t \in \mathbb{R}_+$ on \mathfrak{g} . Then $d^d\xi = t^{d-1}J(\hat{\xi})dt \wedge d\hat{\xi}$, where $d\hat{\xi}$ is the solid angle element and $J(\hat{\xi})$ is the appropriate Jacobian. By the previous two lemmas and the fact that $f \in C^\infty(M)$ implies f is bounded on M , there is some $b' > 0$ such that for R and k sufficiently large

$$\begin{aligned} \left| \int_{\mathfrak{g} \setminus B_R(0)} f \tau \exp \left\{ - \int_{\gamma_\xi} 2k \phi_\xi \right\} d^d\xi \right| \\ = \left| \int_R^\infty \int_{S^{d-1}} \tau(t\hat{\xi}, x_0) \exp \left\{ - \int_{\gamma_{t\hat{\xi}}} 2k \phi_{t\hat{\xi}} \right\} t^{d-1} J(\hat{\xi}) d\hat{\xi} dt \right| \\ \leq b' \int_R^\infty \int_{S^{d-1}} t^{-d} e^{at} e^{-2kCt} t^{d-1} J(\hat{\xi}) d\hat{\xi} dt. \end{aligned}$$

For k sufficiently large there is some $b'' > 0$ such that

$$\int_R^\infty \int_{S^{d-1}} \frac{e^{(a-2kC)t}}{t} J(\hat{\xi}) dt \wedge d\hat{\xi} = \text{vol}(S^{d-1}) \int_R^\infty \frac{e^{(a-2kC)t}}{t} dt \leq b'' e^{-RDk}.$$

Letting $b = b'b''$, we obtain our desired result. \square

5.2 Laplace's approximation

We now turn to the issue of uniformity in Laplace's approximation. We need to show that when applied to the expressions for $I_k(f)([x_0])$ and $J_k(f)([x_0])$, we obtain limits that are uniform in x_0 . To do this, follow a standard argument for Laplace's approximation using the Morse Lemma, but we use a parameterized version (the Morse–Bott Lemma). Then if we simply are careful to bound our errors at each stage, we will obtain the desired uniform limits.

We now state the Morse–Bott Lemma. (A careful proof can be found in [BaH04].) A point $p \in M$ is a critical point of a function $\rho : M \rightarrow \mathbb{R}$ if the differential of ρ at p is zero. A function ρ is said to be a Morse–Bott function if $\text{Crit}(\rho)$, the set of critical points of ρ , is a disjoint union of connected smooth submanifolds and for each critical point, the Hessian $H(\rho)(p)$, in the directions normal to C , is non-degenerate.

Lemma 5.8 (Morse–Bott Lemma). *Let $\rho : M \rightarrow \mathbb{R}$ be a Morse–Bott function, C a connected component of $\text{Crit}(\rho)$ of dimension $2n - d$, and $p \in C$. Then there exists an open neighborhood U of p and a smooth chart $\varphi : U \rightarrow \mathbb{R}^d \times \mathbb{R}^{2n-d}$ such that*

1. $\varphi(p) = 0$
2. $\varphi(U \cap C) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^{2n-d} : x = 0\}$ and
3. $(\rho \circ \varphi^{-1})(z, y) = \rho(C) - \frac{1}{2}(z_1^2 + z_2^2 + \cdots + z_k^2) + \frac{1}{2}(z_{k+1}^2 + \cdots + z_{2n-d}^2)$
 where $k \leq 2n-d$ is the index of $H(\rho)(p)$ and $f(C)$ is the common value of ρ on C . Moreover, if φ' is another smooth chart satisfying (1) and (2) near p , the Jacobian of the coordinate change in the \mathbf{z} -directions, evaluated at p , is $\det \frac{dz}{dz'} = \sqrt{\det H(\rho)(p)}$.

To apply Laplace's approximation in our case, we need to know the value of the density $\tau(\xi, x_0)$ when $\xi = 0$.

Lemma 5.9. *The function τ equals $\text{vol}(G \cdot x_0)$ on the zero-set of the moment map.*

Proof. Let $W_x = \{X_x^\xi : \xi \in \mathfrak{g}\}$. Then we claim that the complexified tangent space to M , at a point $x_0 \in \Phi^{-1}(0)$, decomposes as a B -orthogonal direct sum

$$T_{x_0}^{\mathbb{C}} M = W_{x_0}^{\mathbb{C}} \oplus JW_{x_0}^{\mathbb{C}} \oplus (T_{x_0}^{1,0} M \cap T_{x_0} \Phi^{-1}(0)). \quad (5.5)$$

To see this, first note that by the definition of the $G_{\mathbb{C}}$ -action, the tangent space to the $G_{\mathbb{C}}$ -orbit $G_{\mathbb{C}} \cdot x$ is $W_x \oplus JW_x$. Since $G \cdot x_0$ is a totally real submanifold,

$$W_{x_0} \cap T_{x_0}^{1,0} M = \{0\},$$

and so a dimension count shows that the complexified tangent space decomposes into the claimed direct sum.

We need to show that the direct sums are in fact B -orthogonal. First, if $v \in T_{x_0} \Phi^{-1}(0)$, then $v(\phi_\xi) = 0$ since the moment map is constant (and in fact identically zero) on $\Phi^{-1}(0)$. Hence $B(v, JX^\xi) = \omega(v, X^\xi) = -d\phi_\xi(v) = -v(\phi_\xi) = 0$ and so JW_{x_0} is B -orthogonal to $T_{x_0} \Phi^{-1}(0)$. Next, suppose $v \in (T_{x_0}^{1,0} M \cap T_{x_0} \Phi^{-1}(0))$. Then $Jv = iv$ so that

$$B(X^\xi + JX^{\xi'}, v) = i\omega(X^\xi, v) + i\omega(JX^{\xi'}, v) = iv(\phi_\xi) + v(\phi_\xi) = 0$$

since v is a linear combination of vectors in $T_{x_0} \Phi^{-1}(0)$. This shows $W_{x_0}^{\mathbb{C}} \oplus JW_{x_0}^{\mathbb{C}}$ is B -orthogonal to $(T_{x_0}^{1,0} M \cap T_{x_0} \Phi^{-1}(0))$. Finally, $B(X^\xi, JX^{\xi'}) = \omega(X^\xi, X^{\xi'}) = \{\phi_\xi, \phi_{\xi'}\} = \phi_{[\xi, \xi']} = 0$ on the zero-set, so W_{x_0} is B -orthogonal to JX_{x_0} .

Having established (5.5), we now turn to the computation of τ . On the zero-set of the moment map, the tangent space to the orbit $\exp(i\mathfrak{g}) \cdot x_0$ is JW_{x_0} , which, by (5.5), is B -orthogonal to the tangent space of the zero-set. So if we choose coordinates $\{x^j\}$ on M near $x_0 \in \Phi^{-1}(0)$ such that $\exp(i\mathfrak{g}) \cdot x_0 = \{x^{d+1} = \cdots = x^{2n} = 0\}$, and $x^j(e^{i\xi} \cdot x_0) = \langle \xi_j, \xi \rangle$ (i.e., x^j is the image in M of the linear coordinate in the direction of ξ_j on \mathfrak{g}), then B is block diagonal and so for $x \in \Phi^{-1}(0)$

$$\det B_x = (\det B_x)|_{\exp(i\mathfrak{g}) \cdot x_0} (\det B_x)|_{\Phi^{-1}(0)}.$$

By the standard formula for the volume form in local coordinates, we have

$$\begin{aligned} (\varepsilon_\omega)_x &= d\text{vol}(M)_x = \sqrt{\det B_x} dx^1 \wedge \cdots \wedge dx^{2n} \\ &= \sqrt{\det B_x|_{\exp(i\mathfrak{g}) \cdot x_0}} dx^1 \wedge \cdots \wedge dx^d \wedge \sqrt{\det B_x|_{\Phi^{-1}(0)}} dx^{d+1} \wedge \cdots \wedge dx^{2n} \\ &= \sqrt{\det B_x|_{\exp(i\mathfrak{g}) \cdot x_0}} d^d \xi \wedge d\text{vol}(\Phi^{-1}(0))_{x_0}. \end{aligned}$$

We can rewrite $\det B_x|_{\exp(i\mathfrak{g}) \cdot x_0} = \det \left(B \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) \right)$ in the desired form by noting that $\frac{\partial}{\partial x^j} = \frac{d}{dt}|_{t=0} e^{it\xi_j} x = JX^{\xi_j}$ so that

$$\det \left(B \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) \right) = \det (B (JX^{\xi_j}, JX^{\xi_k})) = \det (B (X^{\xi_j}, X^{\xi_k})) = \det_{\Xi} B.$$

The lemma now follows from Lemma 3.4. \square

We are now ready to perform a uniform Laplace's approximation. We follow a standard argument (see, for example, [BIH75, Sec. 8.3]), although we keep careful track of the error terms to prove uniformity of the limit.

Denote the ball of radius $R > 0$ centered at $0 \in \mathfrak{g}$ by $B_R(0)$.

Lemma 5.10. *Let $f \in C^\infty(M)$ and define*

$$I_{k,R}(f)(x_0) = (k/2\pi)^{d/2} \text{vol}(G \cdot x_0) \int_{B_R(0)} f(\xi, x_0) e^{-k\rho(\xi, x_0)} \tau(\xi, x_0) d^d \xi$$

where $\rho(\xi, x_0) = 2 \int_0^1 \phi_\xi(e^{it\xi} \cdot x_0) dt$. Then there exists some $R > 0$ such that

$$\lim_{k \rightarrow \infty} \left| I_{k,R}(f)(x_0) - 2^{-d/2} \text{vol}(G \cdot x_0) f(x_0) \right| = 0$$

uniformly on $\Phi^{-1}(0)$, where $H(\rho)$ is the Hessian of ρ .

Proof. Laplace's method is based on an application of the Morse–Bott lemma; the zero-set $\{0\} \times \Phi^{-1}(0)$ is a critical submanifold for the function ρ —in fact, $\rho(0, \Phi^{-1}(0)) = 0$ is a minimum since $e^{it\xi}x_0$ is the gradient flow line of ϕ_ξ —and so there exist local coordinates (\mathbf{z}, \mathbf{y}) in some neighborhood of x_0 such that

$$(\rho \circ \varphi^{-1})(\mathbf{z}, \mathbf{y}) = \frac{1}{2} \mathbf{z} \cdot \mathbf{z}. \quad (5.6)$$

For each x_0 the coordinates (\mathbf{z}, \mathbf{y}) are defined in some ball of positive radius; since the zero-set $\{0\} \times \Phi^{-1}(0)$ is compact we can define $R > 0$ to be the minimum of these radii. If we regard the coordinates \mathbf{z} as new coordinates on \mathfrak{g} then there is a Jacobian $J(\mathbf{z}) = \det \frac{\partial \xi}{\partial \mathbf{z}}$ of the coordinate change in the \mathfrak{g} -directions, whence

$$I_{k,R}(f)(x_0) = (k/2\pi)^{d/2} \text{vol}(G \cdot x_0) \int_{B_R(0)} e^{-\frac{k}{2} \mathbf{z}^2} f(\mathbf{z}, x_0) \tau(\mathbf{z}, x_0) J(\mathbf{z}) d^d \mathbf{z}.$$

The exact first-order Taylor series of the function $T(\mathbf{z}, x_0) = f(\mathbf{z}, x_0) \tau(\mathbf{z}, x_0) J(\mathbf{z})$ is given by expanding the identity $T(\mathbf{z}, x_0) = \int_0^1 \frac{d}{dt} T(t\mathbf{z}, x_0) dt$ to yield $T(\mathbf{z}, x_0) = T(0, x_0) + \mathbf{z} \cdot \mathbf{S}(\mathbf{z}, x_0)$, where $S_j(\mathbf{z}, x_0) = \int_0^1 \frac{\partial}{\partial z^j} T(t\mathbf{z}, x_0) dt$. We now have

$$(2\pi/k)^{d/2} \frac{I_{k,R}(x_0)}{\text{vol}(G \cdot x_0)} = T(0, x_0) \int_{B_R(0)} e^{-\frac{k}{2} \mathbf{z}^2} d^d \mathbf{z} + \int_{B_R(0)} e^{-\frac{k}{2} \mathbf{z}^2} (\mathbf{z} \cdot \mathbf{S}) d^d \mathbf{z}.$$

Let $\vec{\nabla} = (\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^d})$ be the vector calculus gradient. Then

$$-\frac{k}{2} e^{-\frac{k}{2} \mathbf{z}^2} (\mathbf{z} \cdot \mathbf{S}) = \vec{\nabla} \cdot (\mathbf{S} e^{-\frac{k}{2} \mathbf{z}^2}) - (\vec{\nabla} \cdot \mathbf{S}) e^{-\frac{k}{2} \mathbf{z}^2}.$$

Substituting this into the last term above and applying the divergence theorem yields

$$\int_{B_R(0)} \vec{\nabla} \cdot (\mathbf{S} e^{-\frac{k}{2} \mathbf{z}^2}) d^d \mathbf{z} = \int_{\partial B_R(0)} (\hat{\mathbf{n}} \cdot \mathbf{S}) e^{-\frac{k}{2} \mathbf{z}^2} d^{d-1} \Omega \leq e^{-\frac{k}{2} R^2} \int_{\partial B_R(0)} (\hat{\mathbf{n}} \cdot \mathbf{S}) d^d \mathbf{z}$$

where $\hat{\mathbf{n}}$ is an outward pointing unit normal vector to $\partial B_R(0)$ and $d^{d-1} \Omega$ is the solid angle element. The last integral above is a continuous function of $x_0 \in \Phi^{-1}(0)$, so it is bounded by a constant $Q_1 > 0$. Similarly, there is some $Q_2 > 0$ such that

$$Q_2 = \max_{\substack{\mathbf{z} \in B_R(0) \\ x_0 \in \Phi^{-1}(0)}} \left| \vec{\nabla} \cdot \mathbf{S} \right|.$$

We can now estimate that for any $x_0 \in \Phi^{-1}(0)$

$$\left| (2\pi/k)^{d/2} \frac{I_{k,R}(x_0)}{\text{vol}(G \cdot x_0)} - T(0, x_0) \int_{B_R(0)} e^{-\frac{k}{2} \mathbf{z}^2} d^d \mathbf{z} \right| \leq \frac{Q_2}{k} \int_{B_R(0)} e^{-\frac{k}{2} \mathbf{z}^2} d^d \mathbf{z} + \frac{Q_1}{k} e^{-\frac{k}{2} r^2}.$$

It is a standard result [BIH75, Lemma 8.3.1] that for any bounded domain D and for every $m > 0$, there exists a constant $C_m > 0$ such that

$$\int_D e^{-\frac{k}{2} \mathbf{z}^2} d^d \mathbf{z} \leq \left(\frac{2\pi}{k} \right)^{d/2} + C_m \lambda^{-m}.$$

Putting this all together, we have for every $x_0 \in \Phi^{-1}(0)$ and for every $m > 0$

$$\begin{aligned} & \left| (2\pi/k)^{d/2} \frac{I_{k,R}(x_0)}{\text{vol}(G \cdot x_0)} - T(0, x_0) \right| \\ & \leq \left| (2\pi/k)^{d/2} I_{k,R}(x_0) - T(0, x_0) \int_{B_R(0)} e^{-\frac{k}{2} \mathbf{z}^2} d^d \mathbf{z} \right| \\ & \quad + |T(0, x_0)| \left| \int_{B_R(0)} e^{-\frac{k}{2} \mathbf{z}^2} d^d \mathbf{z} - (2\pi/k)^{d/2} \right| \\ & \leq \frac{Q_2}{k} (2\pi/k)^{d/2} + \frac{Q_1}{k} e^{-\frac{k}{2} r^2} + (Q_2 + Q_3) C_m k^{-m} \end{aligned}$$

where $Q_3 = \max_{x_0 \in \Phi^{-1}(0)} |T(0, x_0)|$.

All that remains is to note that by the Morse–Bott lemma and Lemma 5.9,

$$T(0, x_0) = \frac{f(x_0) \text{vol}(G \cdot x_0)}{\sqrt{\det H(\rho)(0, x_0)}}.$$

By Theorem 4.1 and Lemma 3.4, the denominator is $2^{d/2} \text{vol}(G \cdot x_0)$. Putting this into the above inequality and taking the limit $k \rightarrow \infty$ we obtain our desired result. \square

If we include the metaplectic correction, then a similar proof applies if we replace $\tau(\xi, x_0)$ by $\tau(\xi, x_0) \exp \left\{ - \int_{\gamma_\xi} \frac{\mathcal{L}_{IX^\xi \varepsilon_\omega}}{2\varepsilon_\omega} \right\}$. Since the argument of the exponent is zero on the zero-set, we obtain a similar result:

Lemma 5.11. *Let $f \in C^\infty(M)$ and define*

$$J_{k,R}(f)(x_0) = (k/2\pi)^{d/2} \int_{B_R(0)} e^{-k\rho(\xi, x_0)} f(\xi, x_0) \exp \left\{ - \int_{\gamma_\xi} \frac{\mathcal{L}_{JX^\xi \varepsilon_\omega}}{2\varepsilon_\omega} \right\} \tau(\xi, x_0) d^d \xi.$$

Then

$$\lim_{k \rightarrow \infty} |J_{k,R}(x_0) - f(x_0)| = 0$$

uniformly on $\Phi^{-1}(0)$.

We are now ready to prove our main result.

Proof of Theorem 5.1. We write I_k as the sum of an integral over $B_R(0)$ and an integral over the complement of $B_R(0)$. Combining Theorem 4.1 and Lemmas 5.9 and 5.10, the first term approaches $2^{-d/2} f(x_0) \text{vol}(G \cdot x_0)$ uniformly as $k \rightarrow \infty$. By Lemma 5.5 the second term approaches 0 uniformly as $k \rightarrow \infty$. If we include the metaplectic correction, the proof is similar, except the first term approaches $f(x_0)$ uniformly. \square

6 Discussion and examples

We begin by noting that compactness is not essential to the issues considered in this paper. Given a noncompact Kähler manifold M acted on by a compact group G , one can define a natural map between the G -invariant holomorphic sections of the relevant line bundle over M and the space of all holomorphic sections of the “quotient” bundle over $M//G$. (Simply restrict the section over M to $\Phi^{-1}(0)$ and then let it descend to $M//G = \Phi^{-1}(0)/G$.) Although it is unlikely that this map is invertible for arbitrary noncompact M , it is likely to be invertible in many interesting examples, and similarly in the presence of the metaplectic correction. It therefore makes sense to investigate the issue of unitarity at least in the more favorable noncompact examples.

Indeed, it even makes sense, to some extent, to consider quantization and reduction for certain *infinite-dimensional* Kähler manifolds. This sort of problem arises naturally in quantum field theories, where the holomorphic approach to quantization is often the most natural one and where one usually has to reduce by an (infinite-dimensional) group of gauge symmetries. Of course, it inevitably requires some creativity to give a sensible meaning to the quantization and to the reduction in infinite-dimensional settings. Nevertheless, there are some interesting examples (discussed below) where this can be done.

In the rest of this subsection, we discuss some noncompact (and, in some cases, infinite dimensional) examples in which the issue of unitarity in quantization versus reduction is of interest. In some of these cases, a Guillemin–Sternberg-type map (with the metaplectic correction) actually turns out to be exactly unitary.

Our first example is the quantization of Chern–Simons theory, as considered in the paper [ADPW91] of Axelrod, della Pietra, and Witten. The authors perform a Kähler quantization of the moduli space \mathcal{M} of flat connections modulo gauge transformations over a Riemann surface Σ . Much of the analysis is done by regarding this moduli space as the symplectic quotient of the space \mathcal{A} of all connections by the group \mathcal{G} of gauge transformations, using the result of Atiyah and Bott that the moment map for the action of \mathcal{G} is simply the curvature. The main result of [ADPW91] is the construction of a natural projectively flat connection that serves to identify the Hilbert spaces obtained by using different complex structures on Σ . The existence of the projectively flat connection shows that the quantization procedure is independent of the choice of complex structure on Σ , since it allows one to identify (projectively) all the different Hilbert spaces with one another.

There is, however, one important issue that is not fully resolved in [ADPW91], namely the issue of the unitarity of the connection. The projectively flat connection “upstairs” on \mathcal{A} is unitary, at least formally. The authors suggest, then, that one should simply define the norm of a section downstairs to be the norm of the corresponding section upstairs, in which case, the connection would (formally) be unitary. However, because \mathcal{A} is infinite-dimensional, it is not entirely clear that this prescription makes sense. Now, *if* it were true that the Guillemin–Sternberg map was unitary in this context, that would mean that computing the norm upstairs is the same as computing the norm downstairs. In that case, one would expect to have unitarity using the natural downstairs norm on the space of sections. Since (as we show in this paper) one cannot expect the Guillemin–Sternberg map to be unitary in general, it is not clear what norm one should use in order to have the projectively flat connection be unitary. Thus, the failure of unitarity (in general) has consequences in this case.

A second example is the canonical quantization of $(1+1)$ -dimensional Yang–Mills theory. In this case, the upstairs space is an infinite-dimensional affine space, namely the cotangent bundle of the space of connections over the spatial circle. Because the upstairs space is an affine space, there is a well-defined (Kähler) quantization, namely a Segal–Bargmann space over an infinite-dimensional vector space (as in [BSZ92]). Unfortunately, as is often the case in field theories, there are no nonzero vectors in this space that are invariant under the action of the gauge group [DH00]. Thus, if one wants to do quantization first and then reduction, some regularization procedure must be used when performing the reduction. Two different regularization procedures have been considered, that of Wren [LW97],[Wre98] (using Landsman’s generalized Rieffel induction [Lan95])

and that of Driver–Hall [DH99] (using a Gaussian measure of large variance to approximate the nonexistent Lebesgue measure). The two procedures give the same result, that the “first quantize and then reduce” space can be identified with a certain L^2 -space of holomorphic functions over the complexified structure group $G_{\mathbb{C}}$.

Meanwhile, the results of [Hal02] indicate that the same L^2 -space of holomorphic functions can be obtained by geometric quantization of $G_{\mathbb{C}}$, provided that the metaplectic correction is used. This means that in this case, a Guillemin–Sternberg-type map, with metaplectic correction, does turn out to be unitary. See also [Hal01].

A third example, related to the second one, is reduction from $G_{\mathbb{C}}$ (which is naturally identified with the cotangent bundle $T^*(G)$) to $G_{\mathbb{C}}/H_{\mathbb{C}}$ (identified with $T^*(G/H)$). Here H is the fixed-point subgroup of an involution, which means that G/H is a compact symmetric space. Results of Hall [Hal02] and Stenzel [Ste99] show that the “first quantize and then reduce” space can be identified with an L^2 -space of holomorphic functions on $G_{\mathbb{C}}/H_{\mathbb{C}}$ with respect to a certain heat kernel measure. (This result can also be obtained from the computation of the relevant orbital integral by Flensted-Jensen [FJ78, Eq. 6.20].) If G/H is again isometric to a compact Lie group (i.e., if $G = U \times U$ and H is the diagonal copy of U), then the above-mentioned results show that the “first reduce and then quantize” can be identified in a natural unitary fashion with the “first quantize and then reduce” space, provided that the metaplectic correction is included in both constructions. So this situation provides another example in which a Guillemin–Sternberg-type map (with metaplectic correction) is unitary.

On the other hand, if G/H is not isometric to a compact Lie group, then the norms on the two spaces are not the same. Rather, the measure obtained in the “first reduce and then quantize” space is the leading term in the asymptotic expansion of the heat kernel measure in the “first quantize and then reduce” space; see the discussion on pp. 244–245 of [Hal02]. (In the case that G/H is isometric to a compact Lie group, the relevant heat kernel is actually equal to the leading term in the expansion, up to an overall constant.) In this case, we see that, even with the metaplectic correction, the measures used to compute the two norms are not equal; they are, however, equal to leading order in \hbar .

A final example, considered in [FMMN03], is reduction from $T^*(G)$ to $T^*(G/\text{Ad}G)$. We may identify $G/\text{Ad}G$ with T/W , where T is a maximal torus in G and W is the Weyl group. Although in this case the action of G is generically nonfree, one should be able to construct a natural map (with the metaplectic correction) by a construction similar to the one we give in this paper. We expect that this map will turn out to be the one given by $F \rightarrow \sigma F|_T$ in the notation of [FMMN03]. If this is the case, then Theorems 2.2 and 2.3 of [FMMN03] will show that the identification of the “first quantize and then reduce space” with the “first reduce and then quantize” space is unitary, up to a constant. The results of [FMMN03] are related to the genus-one case of the quantization of Chern–Simons theory considered in [ADPW91].

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